Reconstruction of ensembles of coupled time-delay systems from time series

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We propose a method to recover from time series the parameters of coupled time-delay systems and the architecture of couplings between them. The method is based on a reconstruction of model delay-differential equations and estimation of statistical significance of couplings. It can be applied to networks composed of nonidentical nodes with an arbitrary number of unidirectional and bidirectional couplings. We test our method on chaotic and periodic time series produced by model equations of ensembles of diffusively coupled time-delay systems in the presence of noise, and apply it to experimental time series obtained from electronic oscillators with delayed feedback coupled by resistors.

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I. INTRODUCTION

Ensembles of coupled delay-differential equations are widely used for modeling and description of processes in various physical [1–3], chemical [4], and biological [5,6] networks with a time-delayed feedback. In order to model these networks from experimental time series it is important to reconstruct the parameters of each system and the architecture of connections between them. Both problems are nontrivial, since even a simple single time-delay system can exhibit high-dimensional chaotic dynamics, and a direct reconstruction of such systems using conventional time-delay embedding techniques often fails. Many of the more advanced methods are based on the projection of an infinite-dimensional phase space of time-delay systems onto low-dimensional subspaces [7–11]. Different criteria are applied to evaluate the quality of the reconstruction, for example, the minimal forecast error of the constructed model [7–9], minimal value of information entropy [10], or various measures of complexity of the projected time series [11]. Some other different methods to estimate parameters of time-delay systems were suggested in the literature: regression analysis [12,13], statistical analysis of time intervals between extrema in the time series [14], nearest neighbor analysis [15], information-theory approaches [16,17], multiple shooting approach [18], seeker optimization algorithm [19], and adaptive synchronization [20,21]. A separate group of methods for the recovery of time-delay systems is based on the analysis of a system's response to external perturbations [22–25]. However, the majority of these methods can be used only to reconstruct model equations for a single time-delay system.

The reconstruction problem becomes even more difficult in the presence of interactions between the time-delay systems and requires development of alternative methods [26]. The architecture and strengths of connections between the network elements define their possible synchronous behavior [27,28]. The problem of detecting the presence, structure, and characteristics of couplings within the multielement ensembles from time series has attracted a lot of attention in recent years. To solve this problem a variety of methods has been proposed including Granger causality [29,30], phase dynamics modeling [31–34], and adaptive feedback control [35–37]. But in all these studies [29–37], either the ensemble elements were without time-delayed feedback or the delays were assumed to be known in advance.

The problem of simultaneous estimation of the network connectivity and node parameters including the delay time for networks of time-delay systems has been addressed recently in Ref. [38]. However, this problem was solved in the absence of noise under the assumption that node functions are invertible and the initial conditions for unknown delays are chosen in a neighbor set of true values. In this paper we propose a method of reconstructing the parameters of elements as well as the architecture and strengths of couplings in ensembles of coupled time-delay systems, which overcomes all the above mentioned limitations. Our method is based on the reconstruction of model delay-differential equations for the ensemble elements and the diagnostics of statistical significance of couplings.

The paper is organized as follows. Section II contains the method description. First we reconstruct the delay time of each element in the ensemble. Then, other parameters and nonlinear functions of the elements and coupling architecture are reconstructed. In Sec. III the method is applied for the reconstruction of various ensembles of coupled time-delay systems from simulated and experimental time series. In Sec. IV we summarize our results.

II. METHOD DESCRIPTION

Let us consider an ensemble composed of diffusively coupled time-delay systems, each described by the equation

$$\varepsilon_i \dot{x}_i(t) = -x_i(t) + f_i(x_i(t-\tau_i)) + \sum_{j=1(j\neq i)}^{M} k_{i,j}(x_j(t) - x_i(t)), \qquad (1)$$

where $i = 1,\ldots,M$; M is the number of elements in the ensemble; the parameter ε_i characterizes the inertial properties of the ith element; τ_i is the delay time; f_i is a nonlinear function; and $k_{i,j}$ are the coupling coefficients characterizing the strength of influence $j \to i$, i.e., from the jth element to the ith one.

It should be noted that the accuracy of delay time recovery has a greater influence on the quality of the system reconstruction than the accuracy of recovery of other parameters. Even a small error in the delay time estimation, as a rule, leads to incorrect reconstruction of coupling architecture and great errors in estimation of other parameters. That is why we propose an approach to the recovery of the element parameters and architecture of couplings in the ensemble of time-delay systems, which involves two steps. At first we recover the delay time τ_i of each element. Then, knowing τ_i, we reconstruct the parameters ε_i and $k_{i,j}$ and nonlinear functions f_i. Such approach allows us to simplify substantially the problem of reconstruction and to obtain a high accuracy of parameter estimation.

A. Reconstruction of delay time of ensemble elements

In Ref. [14] we have shown that time series of single ($k_{i,j} = 0$) time-delay systems (1) practically have no extrema separated in time by the delay time. If such systems perform chaotic oscillations, the extrema in their time series are located irregularly and the time intervals between these extrema can take different values. Taking into account this feature, a method for the delay time recovery has been proposed based on the statistical analysis of time intervals between extrema in the chaotic time series of time-delay system. Defining, for different values of τ, the number N_i of situations where the points of the time series separated in time by τ are both extremal, we can construct the $N_i(\tau)$ plot and recover the delay time τ_i as the value at which the absolute minimum of $N_i(\tau)$ is observed [14].

Let us consider how the presence of couplings between the time-delay systems influences the efficiency of this method. The action of other elements of the ensemble on the time-delay system under consideration disturbs it and results in the disappearance of some extrema in the system time series and appearance of new ones. This effect is especially pronounced in the case of linear coupling of time-delay systems studied in Ref. [26]. In the case of diffusive coupling between the time-delay systems, the method based on the statistical analysis of extrema in the chaotic time series can also be applied to the delay time recovery. Moreover, it remains efficient under essentially stronger couplings than in the case of linear coupling. To explain this feature we differentiate Eq. (1) with respect to t:

$$\varepsilon_i \ddot{x}_i(t) = -\dot{x}_i(t) + \frac{df_i(x_i(t-\tau_i))}{dx_i(t-\tau_i)}\dot{x}_i(t-\tau_i) + \sum_{j=1(j\neq i)}^{M} k_{i,j}(\dot{x}_j(t) - \dot{x}_i(t)). \quad (2)$$

In the presence of inertial properties ($\varepsilon_i > 0$), which corresponds to real situations, the extrema in $x_i(t)$ are close to quadratic ones and therefore $\dot{x}_i(t) = 0$ and $\ddot{x}_i(t) \neq 0$ at the extremal points. If for $\dot{x}_i(t) = 0$ in a typical case $\ddot{x}_i(t) \neq 0$, then, as can be seen from Eq. (2), for $\varepsilon_i \neq 0$ the condition

$$\frac{df_i(x_i(t-\tau_i))}{dx_i(t-\tau_i)}\dot{x}_i(t-\tau_i) + \sum_{j=1(j\neq i)}^{M} k_{i,j}\dot{x}_j(t) \neq 0 \quad (3)$$

must be fulfilled. The condition (3) can be satisfied only if $\dot{x}_i(t-\tau_i) \neq 0$ or/and

$$\sum_{j=1(j\neq i)}^{M} k_{i,j}\dot{x}_j(t) \neq 0. \quad (4)$$

The condition (4) is never fulfilled in the case of the absence of couplings ($k_{i,j} = 0$) and in the case of strong couplings ensuring the synchronization of elements, which results in $\dot{x}_j(t) = \dot{x}_i(t)$. But we have set $\dot{x}_i(t) = 0$ to derive the condition (3). Hence, in these boundary cases the first term in (3) is not equal to zero. By this is meant that the derivatives $\dot{x}_i(t)$ and $\dot{x}_i(t-\tau_i)$ do not vanish simultaneously, i.e., there must be no extremum in $x_i(t)$ separated in time by τ_i from a quadratic extremum. In the intermediate cases of weak and moderate couplings it is possible to find extrema in $x_i(t)$ separated in time by τ_i. However, the probability of such situation is less than the probability to find a pair of extrema separated in time by $\tau \neq \tau_i$. As the result, the number of extrema separated in time by τ_i will be less than the number of extrema separated in time by other values of τ. Hence, the $N_i(\tau)$ plot will have a minimum at $\tau = \tau_i$. Therefore, the qualitative features of the $N_i(\tau)$ plot are retained for system (1) in a wide range of coupling coefficient values.

In spite of a qualitative character of the method explanation, the numerous results of our method application to various ensembles of coupled model and experimental systems with time-delayed feedback presented in Sec. III indicate that the considered method of delay time estimation is efficient for ensembles of coupled time-delay systems and can be successfully used in practice. Note that this method is quick operating, since it uses only operations of comparing and adding and needs neither ordering of data nor calculation of approximation error or certain measure of complexity of the trajectory.

The method described above assumes there is only one delay time per variable. Nevertheless, it can be applied to ensembles composed of time-delay systems with several coexisting delays. As was shown in Ref. [14], the $N_i(\tau)$ plot constructed from time series of a single time-delay system with two different delay times τ_1 and τ_2 exhibits pronounced minima at $\tau = \tau_1$ and $\tau = \tau_2$. However, these minima are not as deep as in the case of a single delay. As the result, the method is less robust to noise in the case of multiple delays.

If a time-delay system performs periodic oscillations, the considered technique fails because extrema in time series are located regularly. For recovering the delay time in single time-delay systems performing periodic oscillations we recently proposed a method based on the analysis of the system response to external disturbance [25]. If one disturbs the $x_i(t)$ variable of a single time-delay system with a single delay by an external signal $y_i(t)$ having the form of rectangular pulses and construct the cross-correlation function

$$C_i(s) = \frac{\langle|\ddot{y}_i(t)||\ddot{x}_i(t+s)|\rangle}{\sqrt{\langle|\ddot{y}_i(t)|^2\rangle\langle|\ddot{x}_i(t)|^2\rangle}}, \quad (5)$$

where the angular brackets denote averaging over time; then $C_i(s)$ will have a pronounced maximum at $s = \tau_i$. In the case of multiple delays, the plot of $C_i(s)$ shows maxima at s values equal to the delay times of the system [25]. A limitation of this method in comparison with the one described above is the necessity of disturbing the system dynamics. However, the method allows one to use very short and low-amplitude pulses in order to reduce the system disturbance to a minimum.

Let us analyze a possibility of this method application to the recovery of delay times in the ensemble of coupled time-delay systems. For the considered type of the element disturbance by the external signal $y_i(t)$, the model equation has the form

$$\varepsilon_i \dot{x}_i(t) = -x_i(t) + f_i(x_i(t-\tau_i) + y_i(t-\tau_i)) + \sum_{j=1(j\neq i)}^{M} k_{i,j}(x_j(t) - x_i(t)). \quad (6)$$

The disturbance $y_i(t)$ has the form of rectangular pulses with amplitude A_i, period T_i, and duration D_i. To recover the delay time τ_i of only the ith element it is sufficient to disturb only this ith element by the signal $y_i(t)$. As we have mentioned above, the presence of connections between the time-delay systems results in the disturbances of their trajectories. These disturbances decrease the sensitivity of the cross-correlation function (5) as the measure for the delay time estimation. As a result, for the recovery of τ_i in the general case it is necessary to increase the amplitude A_i of perturbation in comparison with the case of uncoupled time-delay systems. The method can be used for any values of coupling coefficients $k_{i,j}$. Furthermore, the method can be applied to systems (1) performing either periodic or chaotic oscillations. One more advantage of this method is that it remains efficient under sufficiently high levels of noise, which are several times greater than the noise level allowable for the method of delay time recovery based on the statistical analysis of extrema in time series.

Of course, disturbance of ensemble elements by rectangular pulses is not always possible in practice. However, this kind of time-delay system perturbation is easily realized experimentally, for example, in electronic and radio technical systems [25].

B. Reconstruction of other parameters and architecture of couplings in the ensemble

After reconstruction of τ_i we recover the parameter ε_i, nonlinear function f_i, and coupling coefficients $k_{i,j}$ of the ith time-delay system (1), having at their disposal the time series of oscillations of all elements in the ensemble. To do this, we propose the following approach. Let us write Eq. (1) as

$$\varepsilon_i \dot{x}_i(t) + x_i(t) - \sum_{j=1(j\neq i)}^{M} k_{i,j}(x_j(t) - x_i(t)) = f_i(x_i(t-\tau_i)). \quad (7)$$

If one plots the dependence of the left-hand side of Eq. (7) on $x_i(t-\tau_i)$, it will reproduce the function f_i. Since the parameters ε_i and $k_{i,j}$ are *a priori* unknown, we will search for them by minimizing the function

$$L_i(\varepsilon_i, k_{i,j}) = \sum_{n=1}^{S-1} ((y_{i,n+1} - y_{i,n})^2 + (z_{i,n+1} - z_{i,n})^2), \quad (8)$$

which characterizes the distance between the points in the (y_i, z_i) plane ordered with respect to the coordinate y_i. Here $y_i = x_i(t-\tau_i)$, $z_i = \varepsilon_i \dot{x}_i(t) + x_i(t) - \sum_{j=1(j\neq i)}^{M} k_{i,j}(x_j(t) - x_i(t))$, n is the point serial number, and S is the number of points. In the case of incorrect choice of ε_i and $k_{i,j}$, the points in the (y_i, z_i) plane do not lie on a single-valued curve f_i. Hence, the value of $L_i(\varepsilon_i, k_{i,j})$ will be greater than that for true ε_i and $k_{i,j}$.

We set the initial conditions for ε_i and $k_{i,j}$ and then refine them by the Nelder-Mead method [39] minimizing the function (8), whose minimum is denoted as $L_{i,M}$. At $M \leqslant 4$ and the absence of noise, all parameters are recovered with a high accuracy. However, at $M > 4$ the situation in which the method fails to reveal the nonexisting couplings ($k_{i,j} = 0$) becomes typical. These couplings are detected as weak ones because of indirect couplings via other elements.

To reject insignificant couplings we use the method of successive trial elimination of coefficients $k_{i,j}$ from the model (1). We advance the hypothesis that the coupling $j \to i$ from the jth element to the ith one is absent, eliminate the corresponding coupling coefficient $k_{i,j}$, and reconstruct the other parameters of the model by minimizing the function (8), whose minimum is denoted as $L_{i,j,M-1}$. This procedure is then repeated by eliminating another $k_{i,j}$ at the fixed i, and so on for all $j \neq i$. Note that at each step we assume that the ith element is not affected by only one of the jth elements. Finally, we determine the elimination of which $k_{i,j}$ from (1) yields $L_{i,M-1} = \min_j L_{i,j,M-1}$ and estimate the statistical significance of $L = L_{i,M-1}/L_{i,M}$. In doing this we are guided by the following arguments. At large S, the differences $y_{i,n+1} - y_{i,n}$ and $z_{i,n+1} - z_{i,n}$ in (8) are distributed according to the distribution that is close to the normal one. Here $S/2$ of these differences can be considered as independent because they have no common coordinates. Besides, $L_{i,M}$ depends on M parameters of model (7). This fact reduces the number of independent quantities in (8) to $S/2 - M$. Then, taking into account that a sum of the squares of K independent normal random variables is distributed according to the chi-square distribution with K degrees of freedom [40], we obtain that the $L_{i,M}$ values calculated at different parameters and/or in the presence of noise are distributed according to the chi-square distribution with $S/2 - M$ degrees of freedom and the $L_{i,M-1}$ values are distributed according to the chi-square distribution with $S/2 - M + 1$ degrees of freedom.

If X is a ratio of two independent random variables distributed according to the chi-square distribution with v and w degrees of freedom, respectively, then it has the Fisher-Snedecor distribution with the distribution function

$$F_{v,w}(X) = B_d\left(\frac{v}{2}, \frac{w}{2}\right), \quad (9)$$

where B is the regularized incomplete beta function and $d = vX/(vX + w)$ [41]. Hence, L has the distribution function (9) with $X = L$, $v = S/2 - M + 1$, and $w = S/2 - M$.

We denote the value of L for which $F_{v,w}(L_{1-p}) = 1 - p$, where p is the statistical significance level, as L_{1-p}. Then, if $L > L_{1-p}$, one can conclude at a significance level p that the ith element is affected by all other elements of the ensemble, i.e., all $k_{i,j} \neq 0$. In the opposite case, we conclude that the coupling $j \to i$ between the two corresponding elements is absent and check the significance of other couplings, successively eliminating one by one the remaining links from other elements to the ith one. The procedure is repeated until all couplings become significant. This approach allows one to recover the coupling architecture, parameters of all elements, and nonlinear functions.

If the number of connections between the ensemble elements is known to be small, it is preferable to use the method of successive trial addition of coefficients $k_{i,j}$ to the model for the reconstruction of architecture and strengths of couplings. First we find the minimum $L_{i,1}$ of function (8) assuming that all $k_{i,j}$ are absent in Eq. (1), i.e., there are no couplings. Then we enter one coefficient $k_{i,j}$ into (1) and find the minimum of function (8), which is denoted as $L_{i,j,2}$. This procedure is then repeated by entering another $k_{i,j}$ at the fixed i, and so on for all $j \neq i$. Finally, we determine the entering of which $k_{i,j}$ into the model yields $L_{i,2} = \min_j L_{i,j,2}$. If $L > L_{1-p}$, where $L = L_{i,1}/L_{i,2}$ has the distribution function (9) with $v = S/2 - 1$ and $w = S/2 - 2$, then the introduced coupling is nonzero at a significance level p. The procedure is repeated until the next coupling entered into the model turns out to be insignificant.

III. METHOD APPLICATION

A. Reconstruction of ensemble of coupled Ikeda equations

Let us reconstruct the parameters of elements and coupling architecture in an ensemble of diffusively coupled Ikeda equations described as follows:

$$\dot{x}_i(t) = -x_i(t) + \mu_i \sin(x_i(t-\tau_i) - x_{0i}) + \sum_{j=1(j\neq i)}^{M} k_{i,j}(x_j(t) - x_i(t)). \qquad (10)$$

The Ikeda equation describes a phase lag x of the electrical field across the optical resonator [42]. The parameter μ characterizes the laser power intensity injected into the system, τ is the delay time, and x_0 is the constant phase lag. Equation (10) is a special case of Eq. (1) with $\varepsilon_i = 1$.

As the first example, we consider the simple case of ring unidirectional coupling [Fig. 1(a)]. The chain consists of $M = 10$ nonidentical elements with a boundary condition $x_1 = x_{M+1}$. The parameters of elements are assigned the arbitrary values in the following ranges: $\tau_i \in [2,5]$, $\mu_i \in [15,25]$, $x_{0i} \in [0,2\pi]$, and $k_{i,i-1} \in [0.1,0.5]$; $k_{i,j} = 0$, $j \neq i-1$. In this case all elements exhibit chaotic oscillations. Besides, each element is affected by independent Gaussian noise $\xi_i(t)$ with a zero mean and variance $\sigma_i^2 = 0.01$. For the fixed values of element parameters we generate a collection of 20 time realizations for the ensemble with different initial conditions and realizations of noise. From each time series in this collection we recover the parameters and nonlinear functions of the ensemble elements. As the result, we obtain 20 estimates for each parameter and define their mean, minimal, and maximal values and standard deviation.

We illustrate the results of the method application to the parameter reconstruction of one of the ring elements with the parameters $\tau_7 = 2.15$, $\mu_7 = 21.67$, $x_{07} = 3.88$, and $k_{7,6} =$

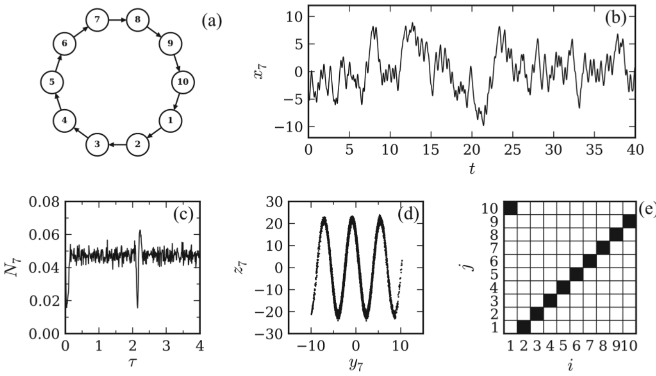

FIG. 1. (a) Chain of ten unidirectionally coupled elements closed in a ring. (b) The chaotic time series of $x_7(t)$ in the ring of Ikeda equations in the presence of Gaussian noise with $\sigma_i^2 = 0.01$. (c) Dependence $N_7(\tau)$. $N_{7\,\min}(\tau) = N_7(2.15)$. (d) Function f_7 recovered in the plane (y_7,z_7), where $y_7 = x_7(t-\tau_7')$ and $z_7 = \dot{x}_7(t) + x_7(t) - \sum_{j=1(j\neq 7)}^{10} k_{7,j}'(x_j(t) - x_7(t))$. (e) Diagram for coupling estimation results shows detected existing couplings (black squares) and undetected nonexisting couplings (white squares).

0.284. Part of its time series is shown in Fig. 1(b). For various τ values we count the number N_7 of situations where $\dot{x}_7(t)$ and $\dot{x}_7(t-\tau)$ are simultaneously equal to zero and construct the $N_7(\tau)$ plot [Fig. 1(c)]. In Fig. 1(c) the step of τ variation is equal to the integration step $h = 0.01$ and $N_7(\tau)$ is normalized to the total number of extrema in the time series. The time derivatives $\dot{x}_7(t)$ are estimated from the time series by applying a local parabolic approximation. The absolute minimum of $N_7(\tau)$ takes place at the true delay time $\tau = \tau_7 = 2.15$ for each of 20 time series of $x_7(t)$ with different realizations of noise. To construct the $N_7(\tau)$ plot we use 40 000 points of time series, which exhibits about 1600 extrema.

Figure 1(d) presents the nonlinear function f_7 recovered employing the method of successive trial addition of coupling coefficients to the model at $p = 0.05$. The function f_7 is constructed at the recovered parameters $\tau_7' = 2.15$, $k_{7,6}' = 0.276$, and $k_{7,j}' = 0$, $j \neq 6$. The estimates of $k_{7,6}$ obtained from a collection of 20 time series lie within the interval [0.268, 0.294]. The mean value of recovered coefficients $k_{7,6}'$ is 0.283 and the standard deviation is 0.007. For each time series in a collection of ensemble realizations we obtain $k_{7,j}' = 0$, $j \neq 6$.

The recovered nonlinear function in Fig. 1(d) coincides closely with the true function of the Ikeda equation. Its approximation with a harmonic function allows us to obtain the parameter estimation $\mu_7' = 21.80$ and $x_{07}' = 3.97$, as the amplitude and initial phase of harmonic function, respectively. The obtained 20 estimates of μ_7 and x_{07} lie within the intervals [20.99, 22.89] and [3.86, 4.01], respectively. Their mean values are 22.04 and 3.92, respectively, and standard deviations are 0.53 and 0.05, respectively. The parameters and coupling coefficients of other elements are reconstructed in a similar way.

The results of recovery of coupling architecture in the ensemble obtained by application of the method of successive trial addition of coupling coefficients to the model are shown in Fig. 1(e). A square with a horizontal coordinate i and a vertical

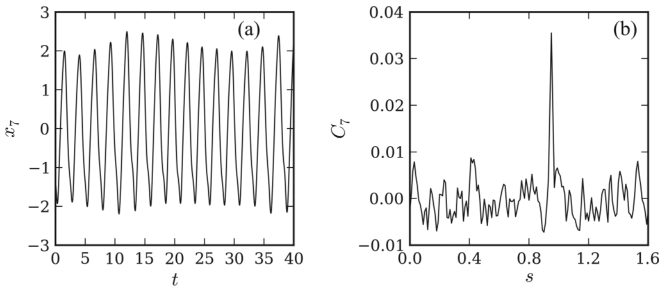

FIG. 2. (a) The time series of $x_7(t)$ in the periodic regime in the presence of Gaussian noise with $\sigma_i^2 = 0.001$. (b) The cross-correlation function $C_7(s)$. $C_{7\,\max}(s) = C_7(0.94)$.

coordinate j shows the influence $j \rightarrow i$, except for the squares in the diagonal, which carry no information. All ten existing couplings are detected at the significance level $p = 0.05$ (black squares). Nonexisting couplings are not detected. The lengths of time series used for constructing Figs. 1(d) and 1(e) are 10 000 points.

In the considered case, the estimated parameters of the ensemble elements are sufficiently close to the true parameters. Note that already small deviations of recovered parameters can result in large biases of the inferred model. However, in the considered example the coupling architecture is accurately recovered from each time series in a collection of ensemble realizations.

Let us consider the case where all elements of the ring perform periodic oscillations. For this purpose the element parameters are assigned the arbitrary values in the following ranges: $\tau_i \in [0.5,1.2]$, $\mu_i \in [4.5,5.5]$, $x_{0i} \in [0,2\pi]$, and $k_{i,i-1} \in [0.1,0.5]$; $k_{i,j} = 0$, $j \neq i-1$. We add an independent Gaussian noise with a zero mean and variance $\sigma_i^2 = 0.001$ into the dynamics of each element. Part of the time series of the element with the parameters $\tau_7 = 0.94$, $\mu_7 = 5.17$, $x_{07} = 3.32$, and $k_{7,6} = 0.284$ is shown in Fig. 2(a).

Since the method of delay time recovery based on the statistical analysis of extrema in time series fails when it is applied to periodic time series, we use the method based on the analysis of the system response to external disturbance. To reconstruct the delay time τ_7 we disturb the $x_7(t)$ variable by a weak external signal $y_7(t)$ having the form of rectangular pulses. Figure 2(b) depicts the cross-correlation function (5) for the case where the pulse signal $y_7(t)$ has the amplitude $A_7 = 0.1$, period $T_7 = 3$, and duration $D_7 = T_7/2$. For the step of s variation equal to 0.01, the plot of $C_7(s)$ exhibits the maximum at $s = 0.94$, i.e., the delay time is recovered accurately for each of 20 time series differing by realizations of noise.

Employing the method of successive trial addition of coupling coefficients to the model we obtain at $p = 0.05$ that the values of μ_7', x_{07}', and $k_{7,6}'$ recovered 20 times under different initial conditions and realizations of noise lie within the intervals [5.09, 6.01], [2.53, 3.87], and [0.260, 0.306], respectively. The mean values of μ_7', x_{07}', and $k_{7,6}'$ are 5.54, 3.21, and 0.279, respectively, and standard deviations are 0.26, 0.36, and 0.011, respectively. For each of 20 cases we obtain $k_{7,j}' = 0$, $j \neq 6$.

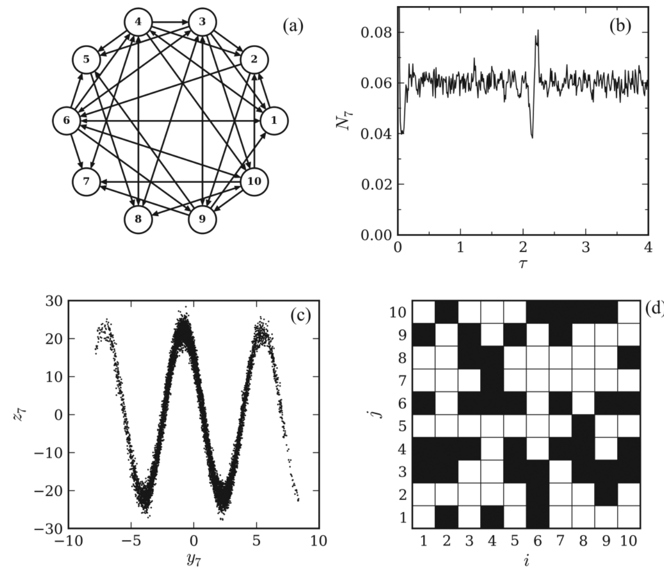

FIG. 3. (a) Coupling architecture in an ensemble of ten elements. (b) Dependence $N_7(\tau)$. $N_{7\,\min}(\tau) = N_7(2.15)$. (c) Function f_7 recovered in the plane (y_7,z_7), where $y_7 = x_7(t-\tau_7')$ and $z_7 = \dot{x}_7(t) + x_7(t) - \sum_{j=1(j\neq 7)}^{10} k_{7,j}'(x_j(t) - x_7(t))$. (d) Diagram for coupling estimation results.

The parameters of other elements are reconstructed in a similar way. The result of recovery of coupling architecture in the chain coincides with the one obtained for the previous example [Fig. 1(e)]. All the couplings are correctly detected at the significance level $p = 0.05$.

The method application is further illustrated for a more complicated coupling architecture. Figure 3(a) shows the coupling architecture generated randomly in an ensemble of ten elements. There are 40 couplings from 90 possible ones. Bidirectional couplings are present along with unidirectional ones. The parameters τ_i, μ_i, and x_{0i} of nonidentical elements are chosen the same as in the first considered example and nonzero coupling coefficients are assigned the arbitrary values in the following range: $k_{i,j} \in [0.1,0.5]$. In this case all the ensemble elements exhibit chaotic oscillations. Each element is affected by independent Gaussian noise with a zero mean and variance $\sigma_i^2 = 0.04$.

We illustrate the results of reconstruction of one of the elements with the parameters $\tau_7 = 2.15$, $\mu_7 = 21.67$, $x_{07} = 3.88$, $k_{7,4} = 0.445$, $k_{7,6} = 0.172$, $k_{7,9} = 0.311$, $k_{7,10} = 0.435$, and $k_{7,j} = 0$, $j = 1,2,3,5,8$. The dependence $N_7(\tau)$ with the step of τ variation equal to 0.01 is presented in Fig. 3(b). The minimum of $N_7(\tau)$ is observed at the true delay time $\tau = \tau_7 = 2.15$ for each of 20 time series of $x_7(t)$ with different realizations of noise.

Figure 3(c) shows the nonlinear function f_7 recovered using the method of successive trial addition of coupling coefficients to the model at $p = 0.05$. The function f_7 is constructed at the recovered parameters $\tau_7' = 2.15$, $k_{7,4}' = 0.517$, $k_{7,6}' = 0.188$, $k_{7,9}' = 0.355$, $k_{7,10}' = 0.490$, and $k_{7,j}' = 0$, $j = 1,2,3,5,8$. The values of $k_{7,4}'$, $k_{7,6}'$, $k_{7,9}'$, and $k_{7,10}'$ estimated from a collection of 20 time series lie within the intervals [0.401, 0.517], [0.142, 0.188], [0.277, 0.355], and [0.396, 0.490], respectively. Their mean values are 0.453, 0.164, 0.313, and 0.441, respectively, and standard deviations are 0.024, 0.011, 0.016, and 0.020, respectively. For each of 20 time series we obtain $k'_{7,j} = 0$, $j = 1,2,3,5,8$.

Inaccurate estimation of coupling coefficients and noise level higher than that in the first example result in the worse quality of function f_7 recovery in comparison with Fig. 1(d). Approximation of the recovered function f_7 with a harmonic function gives us the following parameter estimation: $\mu'_7 = 22.00$ and $x'_{07} = 3.97$. The obtained 20 estimates of μ_7 and x_{07} lie within the intervals [20.66, 23.41] and [3.66, 4.47], respectively. Their mean values are 21.90 and 4.12, respectively, and standard deviations are 0.70 and 0.19, respectively.

Similarly the parameters and coupling coefficients of other elements are reconstructed. The results of recovery of coupling architecture in the ensemble are presented in Fig. 3(d). All 40 existing couplings are detected at the significance level $p = 0.05$ employing either the method of addition of couplings or the method of successive trial elimination of coupling coefficients from the model. The coupling architecture is accurately recovered from each time series in a collection of ensemble realizations.

Let us consider the case where the ensemble elements have the same parameter values and coupling architecture and strengths, but are affected by strong independent Gaussian noise with a zero mean and variance $\sigma_i^2 = 0.36$. At such strong noise, the method of the delay time recovery based on the statistical analysis of extrema in time series fails. To reconstruct the delay time of the ith element we disturb the $x_i(t)$ variable by a weak signal $y_i(t)$ having the form of rectangular pulses. Figure 4(a) depicts the cross-correlation function (5) for the case where the variable $x_7(t)$ is disturbed by the pulse signal $y_7(t)$ having the amplitude $A_7 = 0.15$, period $T_7 = 5$, and duration $D_7 = T_7/2$. For the step of s variation equal to 0.01 the plot of $C_7(s)$ exhibits the maximum at $s = \tau_7 = 2.15$ for each of 20 time series differing by realizations of noise. In spite of the high level of noise, the delay time is recovered accurately for all elements.

The method of addition of couplings, as well as the method of elimination of couplings, gives at $p = 0.05$ the following estimation of the seventh element parameters obtained from a collection of 20 time series: $k'_{7,4}$, $k'_{7,6}$, $k'_{7,9}$, $k'_{7,10}$, μ'_7, and x'_{07} lie within the intervals [0.388, 0.511], [0.113, 0.202], [0.244, 0.364], [0.380, 0.497], [21.01, 24.14], and [3.44, 4.42], respectively. Their mean values are 0.434, 0.157, 0.298, 0.428, 22.56, and 4.10, respectively, and standard deviations are 0.029, 0.022, 0.028, 0.028, 0.78, and 0.25, respectively. The true parameter values are the same as those in the previous example illustrated in Fig. 3.

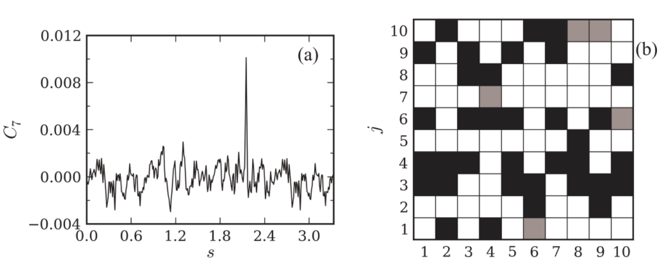

FIG. 4. Case of high level of noise. (a) The cross-correlation function $C_7(s)$. $C_{7\,\max}(s) = C_7(2.15)$. (b) Diagram for coupling estimation results shows detected existing couplings (black squares), undetected nonexisting couplings (white squares), and missed existing couplings (gray squares).

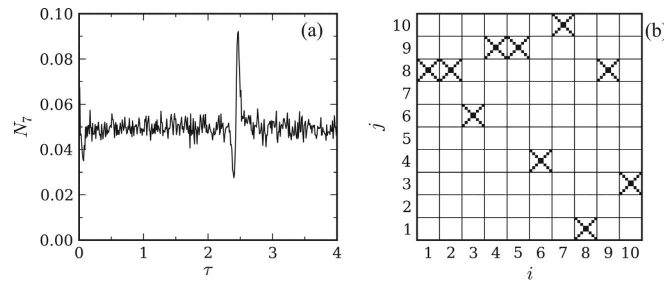

FIG. 5. Case of large number of couplings. (a) Dependence $N_7(\tau)$. $N_{7\,\min}(\tau) = N_7(2.40)$. (b) Diagram for coupling estimation results shows detected existing couplings (white squares) and undetected nonexisting couplings (squares with crosses).

The results of reconstruction of coupling architecture in the ensemble are presented in Fig. 4(b) for the recovered parameters $k'_{7,4} = 0.396$, $k'_{7,6} = 0.142$, $k'_{7,9} = 0.262$, and $k'_{7,10} = 0.392$. We detected 35 couplings from 40 existing ones at the significance level $p = 0.05$. Five existing couplings are missed (gray squares) because of high level of noise. Estimation from other time series gives similar pictures, but the number of missed connections and their locations fluctuate from one set of time series to another one. Which couplings are not detected from the given time series is determined by concrete realizations of noises corresponding to the analyzed time series. To decrease the number of missed couplings, one should specify greater p. However, at that, probability to detect spurious (nonexisting) connections increases.

At last, we consider the case where 80 of 90 possible couplings exist. The element parameters are assigned the arbitrary values in the same ranges as in the above case of 40 couplings. Each element is affected by independent Gaussian noise with a zero mean and variance $\sigma_i^2 = 0.01$.

The location of the absolute minimum of $N_i(\tau)$ allows us to recover the delay time accurately for each element of the ensemble. As an example, Fig. 5(a) shows the dependence $N_7(\tau)$ with the step of τ variation equal to 0.01. The minimum of $N_7(\tau)$ is observed at the true delay time $\tau = \tau_7 = 2.40$ for each of 20 time series. The results of recovery of coupling architecture obtained by application of the method of successive trial elimination of coupling coefficients from the model are presented in Fig. 5(b). For better visual perception we inverted the colors in Fig. 5(b) in comparison with other diagrams. All 80 existing couplings are detected at the significance level $p = 0.05$. Nonexisting couplings are not detected.

B. Reconstruction of ensemble of coupled Mackey-Glass equations

Let us reconstruct the parameters of elements and coupling architecture in an ensemble of diffusively coupled

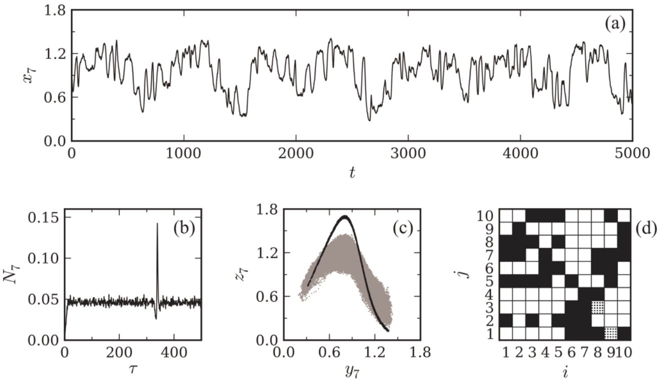

FIG. 6. (a) The time series of $x_7(t)$ in the ensemble of Mackey-Glass equations in the presence of Gaussian noise with $\sigma_i^2 = 10^{-4}$. (b) Dependence $N_7(\tau)$. $N_{7\,\mathrm{min}}(\tau) = N_7(335)$. (c) Function f_7 recovered in the plane (y_7,z_7), where $y_7 = x_7(t-\tau_7')$ and $z_7 = \varepsilon_7'\dot{x}_7(t) + x_7(t) - \sum_{j=1(j\neq 7)}^{10} k'_{7,j}(x_j(t) - x_7(t))$ under the assumption of absence of couplings (gray shading) and presence of couplings (black). (d) Diagram for coupling estimation results shows detected existing couplings (black squares), undetected nonexisting couplings (white squares), and spuriously detected couplings (squares with dots).

Mackey-Glass equations described by Eq. (1) with the function

$$f_i(x_i(t-\tau_i)) = \frac{a_i x_i(t-\tau_i)}{b_i\left(1 + x_i^{10}(t-\tau_i)\right)} \tag{11}$$

and $\varepsilon_i = 1/b_i$. The Mackey-Glass equation models the production of red blood cells and is widely used in simulation of time-delay systems [43].

We consider the case where the ensemble is composed of nonidentical elements whose parameters take arbitrary values in the following ranges: $\tau_i \in [300{,}400]$, $\varepsilon_i \in [8{,}12]$, $a_i \in [0.2{,}0.25]$, and $k_{i,j} \in [0.01{,}0.05]$. In this case all the elements exhibit chaotic oscillations. Besides, each element is affected by independent Gaussian noise $\xi_i(t)$ with a zero mean and variance $\sigma_i^2 = 10^{-4}$.

Part of the time series of the element with the parameters $\tau_7 = 335$, $\varepsilon_7 = 10.2$, $k_{7,1} = 0.011$, $k_{7,2} = 0.046$, $k_{7,3} = 0.043$, $k_{7,4} = 0.016$, and $k_{7,j} = 0$, $j = 5,6,8,9,10$ is shown in Fig. 6(a). To construct the $N_7(\tau)$ plot we use 40 000 points of time series of $x_7(t)$, which exhibits about 2600 extrema. For the step of τ variation equal to 1, the dependence $N_7(\tau)$ shows the minimum at the true delay time $\tau = \tau_7 = 335$ for each of 20 time series [Fig. 6(b)].

Figure 6(c) depicts in gray the function f_7 recovered under the assumption that there are no connections between the ensemble elements. This function is constructed at the recovered parameters $\tau_7' = 335$, $\varepsilon_7' = 8.4$, and $k'_{7,j} = 0$, $j = 1,\ldots,10\,(j \neq 7)$. The function f_7 recovered using the method of successive trial addition of coupling coefficients to the model at $p = 0.05$ is shown in Fig. 6(c) in black. This function is constructed at the recovered parameters $\tau_7' = 335$, $\varepsilon_7' = 10.0$, $k'_{7,1} = 0.012$, $k'_{7,2} = 0.047$, $k'_{7,3} = 0.044$, $k'_{7,4} = 0.017$, and $k'_{7,j} = 0$, $j = 5,6,8,9,10$.

The values of ε_7', $k'_{7,1}$, $k'_{7,2}$, $k'_{7,3}$, and $k'_{7,4}$ estimated from a collection of 20 time series lie within the intervals [9.7, 10.5], [0.006, 0.017], [0.039, 0.054], [0.036, 0.052], and [0.007, 0.020], respectively. Their mean values are 10.1, 0.012, 0.046, 0.045, and 0.014, respectively, and standard deviations are 0.2, 0.003, 0.004, 0.004, and 0.003, respectively.

One can see that the quality of nonlinear function recovery in Fig. 6(c) is much better, if the coupling architecture is taken into account. Inaccuracy of the parameter reconstruction is caused mainly by the presence of noise. As well as in the other considered examples, the lengths of time series used for the recovery of parameters are 10 000 points.

The parameters and coupling coefficients of other elements are reconstructed in a similar way. The results of recovery of coupling architecture in the ensemble obtained by application of the method of addition of couplings are presented in Fig. 6(d). All 40 existing couplings are detected at the significance level $p = 0.05$. Two couplings are detected spuriously (squares with dots). However, this rate of errors lies within acceptable range at the given p. Estimation from other time series gives similar pictures with a slightly fluctuating rate of spurious couplings. Decreasing p, it is possible to avoid spuriously detected couplings. However, at that, the probability to miss existing connections rises. Recovering the coupling architecture from the same time series by application of the method of elimination of couplings, we obtain a greater number of spuriously detected couplings at the same p.

C. Reconstruction of ensemble of coupled experimental electronic oscillators with time-delayed feedback

We apply the method to experimental time series obtained from three coupled electronic oscillators with time-delayed feedback. Figure 7(a) shows a block diagram of the experimental setup involving three coupled oscillators, each comprising a delay line, a nonlinear device, and a low-frequency first-order *RC* filter. The delay lines and nonlinear devices are implemented on microcontrollers, while the filters

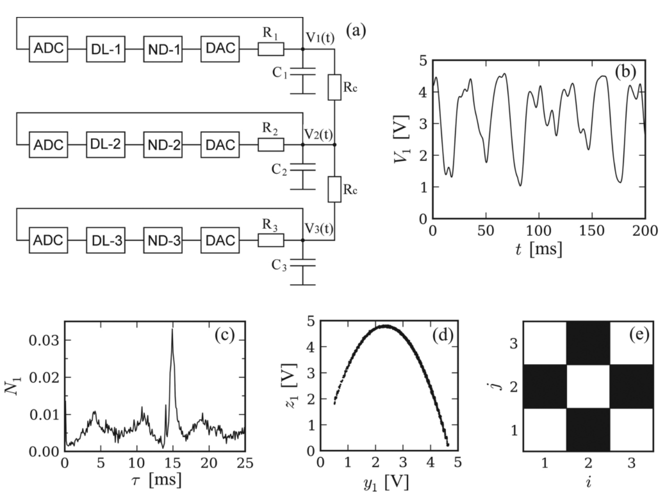

FIG. 7. (a) Block diagram of the experimental setup. DL are the delay lines, ND are the nonlinear devices, ADC are the analog-to-digital converters, and DAC are the digital-to-analog converters. (b) The chaotic time series of $V_1(t)$. (c) Dependence $N_1(\tau)$. $N_{1\,\mathrm{min}}(\tau) = N_1(13.6\text{ ms})$. (d) Function f_1 recovered in the plane (y_1,z_1), where $y_1 = V_1(t-\tau_1')$ and $z_1 = \varepsilon_7'\dot{V}_1(t) + V_1(t) - \sum_{j=2}^{3} k'_{1,j}(V_j(t) - V_1(t))$. (e) Diagram for coupling estimation results.

are analog devices. The digital and analog elements of this scheme are linked via the corresponding analog-to-digital converters (ADCs) and digital-to-analog converters (DACs). The oscillators are coupled via resistors R_c. A model equation for the ith element in the ensemble is as follows:

$$R_i C_i \dot{V}_i(t) = -V_i(t) + f_i(V_i(t-\tau_i)) + \sum_{j=1(j\neq i)}^{M} k_{i,j}(V_j(t) - V_i(t)), \qquad (12)$$

where $V_i(t)$ and $V_i(t-\tau_i)$ are the delay line input and output voltages, respectively, τ_i is the delay time, R_i and C_i are the resistance and capacitance, respectively, and f_i is the transfer function of the nonlinear device. Equation (12) is of form (1) with $\varepsilon_i = R_i C_i$.

All the nonlinear devices have a quadratic transfer function. We record the chaotic signals $V_i(t)$ of three nonidentical oscillators using a three-channel ADC with the sampling frequency $f_s = 10$ kHz. Figure 7(b) shows a part of the time series of $V_1(t)$ in the first oscillator having the parameters $\tau_1 = 13.6$ ms, $\varepsilon_1 = 2.88$ ms, $k_{1,2} = R_1/R_c = 0.1$, and $k_{1,3} = 0$.

For the step of τ variation equal to the sampling time $T_s = 0.1$ ms, the absolute minimum of $N_1(\tau)$ takes place at $\tau = 13.6$ ms [Fig. 7(c)]. The function f_1 recovered from the experimental time series using the method of addition of coupling coefficients to the model at $p = 0.05$ is presented in Fig. 7(d). This function is constructed at the recovered parameters $\tau_1' = 13.6$ ms, $\varepsilon_1' = 2.74$ ms, $k_{1,2}' = 0.098$, and $k_{1,3}' = 0$. It coincides closely with the true transfer function f_1 of the nonlinear device of the first oscillator. The method of elimination of couplings gives the same results.

In a similar way we reconstruct the characteristics of other oscillators. The results of recovery of coupling architecture in the ensemble are shown in Fig. 7(e). All couplings are correctly detected at the significance level $p = 0.05$.

IV. CONCLUSION

We have proposed the method for reconstructing the parameters of elements and architecture of connections in ensembles of coupled time-delay systems from time series. The procedure of reconstruction involves two steps. At first we recover the delay time of each element using either the method based on the statistical analysis of extrema in time series or the method based on the analysis of the system response to weak external disturbance having the form of rectangular pulses. The first of these methods is applicable to chaotic time series weakly corrupted by noise. The second one can be applied to time-delay systems performing both chaotic oscillations (even in the presence of high levels of noise) and periodic oscillations. However, a limitation of the second technique is the necessity of disturbing the system dynamics.

At the second step we reconstruct the other parameters and architecture of couplings using the method based on the reconstruction of model delay-differential equations for the ensemble elements and estimation of statistical significance of couplings. To test the significance of links we employ the method of successive trial elimination or addition of couplings. The method of trial elimination of coupling coefficients from the model is most efficient for the recovery of ensembles, in which the number of existing couplings is much greater than the number of nonexisting couplings. For the recovery of ensembles with a small number of connections between the elements, the method of trial addition of coupling coefficients to the model should be used. In the cases where the numbers of existing and nonexisting couplings are comparable, it is preferable to use the method of addition of couplings. For such ensembles it ensures the same or higher accuracy of reconstruction than the method of elimination of couplings and is more quick operating.

The proposed technique can be applied to ensembles composed of nonidentical time-delay systems with an arbitrary number of unidirectional and bidirectional couplings between them. We verified the method by applying it to chaotic and periodic time series produced by model equations of ensembles of diffusively coupled time-delay systems with a single delay in the presence of noise, and experimental time series obtained from electronic oscillators with delayed feedback coupled by resistors. The method can be applied to ensembles of time-delay systems with multiple delays, but in this case it is less robust to noise.

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