ISSN 1063-7850, Technical Physics Letters, 2014, Vol. 40, No. 10, pp. 954–956. © Pleiades Publishing, Ltd., 2014.
Original Russian Text © E.V. Sidak, D.A. Smirnov, B.P. Bezruchko, 2014, published in Pis'ma v Zhurnal Tekhnicheskoi Fiziki, 2014, Vol. 40, No. 20, pp. 104–110.

Estimation of the Coupling Delay Time from Time Series of Autooscillators with Allowance for the Autocorrelation Function of Phase Noise

E. V. Sidak*, D. A. Smirnov, and B. P. Bezruchko
Institute of Radio Engineering and Electronics (Saratov Branch), Russian Academy of Sciences, Saratov, 410019 Russia
**e-mail: sidakev@gmail.com*

Received June 16, 2014

Abstract—We propose to determine the delay time of coupling between autooscillatory systems using a modification of the well-known method based on a model of the oscillation phase dynamics, which ensures its applicability in the case of nonwhite noises influencing this dynamics. For this purpose, the method of calculating a confidence interval is refined by allowance for the correlation time of the phase noise. The applicability of the proposed estimator is illustrated by the results of numerical experiments with systems of standard coupled oscillators.

DOI: 10.1134/S1063785014100289

The task of revealing delayed coupling between oscillatory systems by using their time series is encountered in various fields of science, including bio- and geophysics (see, e.g., [1, 2]). An important characteristic of this interaction is the coupling delay time, which frequently determines the complexity of observed dynamics. Methods of determining this characteristic have been considered in a number of investigations [3–5]. The good prospects for practical application are related to an interval estimator based on empirical modeling of the observed phase dynamics [6], since this approach is applicable in the case of rather short time series (with a length of several characteristic periods), which has been previously demonstrated for autooscillatory systems with a limit cycle under the action of a normal white noise. However, the properties of phase noises in many cases can differ from this type, for example, in some standard chaotic oscillators, such as the Rössler system [7, 8], and oscillators occurring under the action of a narrow-band random process.

In order to obtain reliable estimates of the coupling delay time in these cases, it is necessary to check the applicability of the proposed method and introduce necessary corrections, which is the main purpose of the present work.

It has been reliably established and grounded [7, 9, 10] that, in description of the dynamics of weakly coupled autooscillatory systems with limit cycle, perturbed by weak noises, it is possible to ignore the influence of amplitudes in the equations of phase evolution. According to the approach proposed and developed in [3, 6] for systems expected to possess these properties, the first step is to use the records of oscillations of the observed variables $x_1(t)$ and $x_2(t)$ for calculating by one of the well-known methods [7] the time series of their phases $\{\phi_1(t_1), \ldots, \phi_1(t_N)\}$ and $\{\phi_2(t_1), \ldots, \phi_2(t_N)\}$, where $t_i = i\Delta t$, Δt is the sampling interval and N is the length of the time series. Then, to determine the influence of the first oscillator on the second one (and vice versa, by analogy), a model of the phase dynamics of the second oscillator is constructed in the form of a stochastic difference equation

$$\phi_2(t+\tau) - \phi_2(t) = F(\phi_2(t), \phi_1(t-\Delta)) + \varepsilon(t), \quad (1)$$

where the phase increments are taken on a time interval of fixed length τ (which is a parameter of the given method), ε is the τ-integrated phase noise, F is a trigonometric polynomial, and Δ is a free parameter of the model. The coefficients of polynomial F are determined by least squares through minimization of $S(\Delta) = \langle \hat{\varepsilon}^2(t_i)\rangle$, where $\hat{\varepsilon}(t_i) = \phi_2(t_i+\tau) - \phi_2(t_i) - F(\phi_2(t_i), \phi_1(t_i - \Delta))$ and angular brackets denote averaging over the observation times t_i. The S value is then minimized with respect to various Δ and the point estimate of the delay time takes the form of $\hat{\Delta} = \Delta_{\min} + \tau/2$, where $\Delta_{\min} = \arg\min_{\Delta} S(\Delta)$. The variance of estimate $\hat{\Delta}$ under the assumption of white noise in the initial system and, hence, of the noise autocorrelation function (ACF) decaying to zero on the interval from 0 to τ, is determined using the formula

$$\hat{\sigma}_\Delta^2 = \frac{2\hat{\sigma}_\varepsilon^2}{N'}\left(\left.\frac{d^2S(\Delta)}{d\Delta^2}\right|_{\Delta=\Delta_{\min}}\right)^{-1}, \quad (2)$$

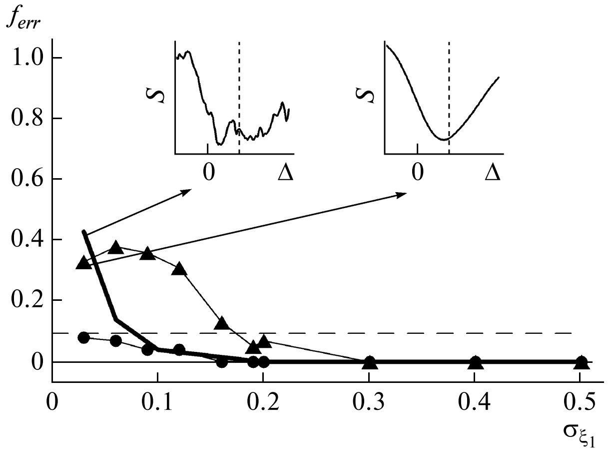

Fig. 1. Determination of the coupling delay time in system (3) as characterized by frequency $f_{\rm err}$ of erroneous conclusions for (▲) estimator (2) (●) estimator (4) with $\alpha_1 = 0.11$, $\alpha_2 = 0.09$, and (thick solid curve) both estimators with $\alpha_1 = 0.11$ and $\alpha_2 = 9$. The horizontal dashed line shows the permissible level of the frequency of errors. The insets show the typical examples of $S(\Delta)$ for individual time series, where vertical dashed lines indicate the true delay time Δ_0.

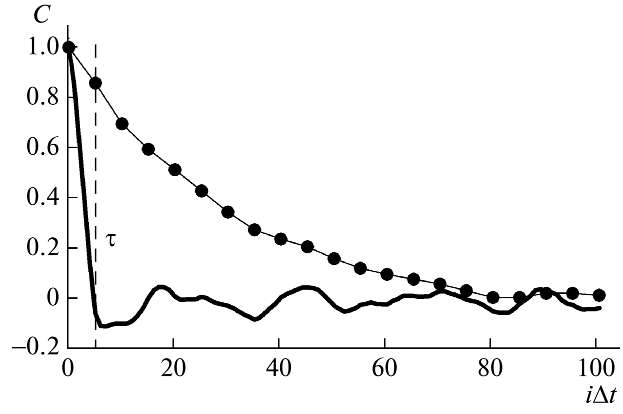

Fig. 2. Determination of the ACF of phase noise in model (1) for (●) $\alpha_1 = 0.11$, $\alpha_2 = 0.09$, $\sigma_{\xi_1} = 0.12$ and (solid curve) $\alpha_1 = 0.11$, $\alpha_2 = 9$, and $\sigma_{\xi_1} = 0.03$. The vertical dashed line indicates the value of τ.

where $N' = N\Delta t/\tau$ is the number of intervals with length τ in the given series and $\hat{\sigma}_\varepsilon^2 = \min_\Delta S(\Delta)$ is the estimate of the variance of noise ε. In order to determine the second derivative in Eq. (2), $S(\Delta)$ in the vicinity of $\Delta_{\min}$ is approximated by a quadratic parabola. The interval estimate of delay time (95% confidence interval) is calculated as $\hat{\Delta} \pm 1.96\hat{\sigma}_\Delta$. If the interval estimate given by formula (2) is smaller than $2\Delta t$, the estimate is taken to be $\hat{\Delta} \pm \Delta t$. The proposed method has proven to be effective for phase oscillators and the der Pol oscillator in the presence of white noise [6].

Let us consider oscillators with noise of other types, e.g., phase oscillators

$$\begin{aligned} d\phi_1(t)/dt &= \omega_1 + \xi_1(t), \\ d\xi_1(t)/dt &= -\sigma_1\xi_1(t) + \eta_1(t), \\ d\phi_2(t)/dt &= \omega_2 + k\sin(\phi_1(t-\Delta_0) - \phi_2(t)) + \xi_2(t), \\ d\xi_2(t)/dt &= -\alpha_2\xi_2(t) + \eta_2(t), \end{aligned} \qquad (3)$$

where $\omega_{1,2}$ are the angular eigenfrequencies, k is the coupling coefficient, Δ_0 is the delay time of $1 \longrightarrow 2$ coupling, $\eta_{1,2}$ is the mutually uncorrelated white noise with autocovariances $\langle\eta_k(t)\eta_k(t')\rangle = D_k\delta(t - t')$, and $\xi_{1,2}$ is color-frequency noise. The variance of ξ_k is expressed through the white-noise intensity as $\sigma_{\xi k}^2 = D_k/(2\alpha_k)$.

An ensemble consisting of 100 pairs of time series was obtained by integrating Eqs. (3) by the Euler method at a 0.01 step on a 0.3 sampling interval (about 20 points over a characteristic period for the parameters specified below) for series length $N = 2094$ (about 100 characteristic periods) and $\tau = 1.5$ (quarter of the period) [6]. Each pair of time series was used to calculate the delay time and frequency $f_{\rm err}$ of erroneous conclusions concerning Δ_0 (i.e., the frequency of situations when the $\Delta 0$ value does not fall in the interval of $\hat{\Delta} \pm 1.96\hat{\sigma}_\Delta$). The proposed method works correctly provided that the probability of erroneous conclusions does not exceed 0.05 (for the 95% confidence interval). With allowance for the fluctuations in $f_{\rm err}$ (distributed according to the Bernoulli law), the admitted $f_{\rm err}$ value for an ensemble of 100 pairs of time series amounts to 0.1.

Figure 1 (triangles) presents results of calculations for $\omega_1 = 1.05$, $\omega_2 = 0.95$, $k = 0.1$, and $\Delta_0 = 12$ (40 data points), $\sigma_{\xi_2} = 0.06$, $\alpha_1 = 0.11$, and $\alpha_2 = 0.09$. The frequency of erroneous conclusions is large for $\sigma_{\xi_1} <$ 0.17. For diagnostics of this situation, let us consider the ACF of noise ε, which was estimated using the residual errors as $C(i\Delta t) = \langle\hat{\varepsilon}(t)\hat{\varepsilon}(t + i\Delta t)\rangle/\hat{\sigma}_\varepsilon^2$ ($i = 1, 2, \ldots$). This value decays to a small value over an interval that is much greater than τ (Fig. 2, circles), in contrast to what would be observed in the case of white noise in the initial equations of system dynamics.

This state of affairs is also possible for low-dimensional nonlinear oscillators in the regime of deterministic chaos, where the use of Eq. (1) (which does not explicitly takes into account the amplitude dynamics) is not strictly justified but frequently provides good approximation (with the properties of phase noise ε determined by the influence of slowly varying chaotic amplitudes [7]). Therefore, it can be expected that the proposed method will also be effective in this case (as a heuristic approach), but only provided that the properties of phase noise ε in Eq. (1) are taken into account properly instead of it being assumed that it is white noise.

With allowance for this, let us estimate the ACF of noise ε by residual errors of the model (Fig. 2) and determine time T of its decay to some small value (as will be shown below, the level of 0.2 gives an acceptable result for time series with a length of 100 characteristic periods). The number of mutually independent ε values over the length of time series can be estimated from, below, $N'' = N\Delta/L$, where $L = \max[T, \tau]$. Then, the estimation of variance takes the form

$$\hat{\sigma}_{\Delta}^2 = \frac{2\hat{\sigma}_{\varepsilon}^2}{N''}\left(\left.\frac{\partial^2 S(\Delta)}{\partial\Delta^2}\right|_{\Delta = \Delta_{\min}}\right)^{-1}, \quad (4)$$

which gives a wider interval than formula (2) for $T > \tau$.

Variance $\sigma_{\xi_1}^2$ of noise in the leading oscillator in system (3) was varied in a broad range by changing D_1 at fixed values of the other parameters. Figure 1 shows a plot of f_{err} versus $\sigma_{\xi_1}^2$ for estimators (2) and (4) at fixed values of α_1, α_2, and $\sigma_{\xi_2}^2$. Figure 2 presents the ACFs of residual errors for model (1) in cases of $\alpha_1 = 0.11$, $\alpha_2 = 0.09$ (long-range correlations) and $\alpha_1 = 0.11$, $\alpha_2 = 9$ (rapidly decaying correlations—i.e., almost white noise). The value of f_{err} for estimator (2) is large at $\alpha_1 = 0.11$, $\alpha_2 = 0.09$, and $\sigma_{\xi_2} = 0.06$ (Fig. 1) provided sufficiently small σ_{ξ_1} (below 0.2). Allowance for the noise correlation time in Eq. (4) decreases the frequency of errors to values not exceeding the permissible level (horizontal dashed line).

It should be noted that, for $\alpha_1 = 0.11$, $\alpha_2 = 0.09$, and small σ_{ξ_1}, the allowance for noise correlations does not help to reduce the frequency of errors because S has poor sensitivity to trial Δ values and random fluctuations shift the point of $S(\Delta)$ minimum. This situation can be recognized based on the $S(\Delta)$ plot, which has no clearly pronounced minimum (Fig. 1, left inset), in contrast to a “favorable” situation (right inset).

This work did not study in detail the method of signal-phase calculation and assumed that the phase values were known, although this issue is worth separate consideration. However, it should be noted that the results obtained for coupled van der Pol oscillators and Rössler systems proved to be analogous to those presented above in a broad range of parameters (including strongly perturbed and even chaotic regimes), where the phases were determined from observed signals by a traditional method [7, 9] based on the Hilbert transform.

Thus, we have proposed a correction to the interval estimate of the coupling delay time between oscillators, which is based on the allowance for the ACF of phase noise in calculations of the confidence interval. The necessity of making this correction for eliminating erroneous conclusions was confirmed by numerical experiments on standard coupled oscillators. The proposed modified method should be used for analysis of nonlinear oscillatory systems in various applications, since even standard mathematical models of oscillators of low dimensionality (such as the Rössler, Lorentz, and some other systems) exhibit phase noises with nontrivial properties [7, 8], including non-Gaussian distribution and long-range correlations. Since the correlation time most significantly influences the informativity of data for statistical evaluation, it can be expected that the proposed correction will be sufficient to provide for reliability of this method in a broad range of practical situations.

Acknowledgments. This study was supported by the Russian Foundation for Basic Research (projects nos. 14-02-31129-mol-a and 14-02-00492-a), the Presidential Program of Support for Leading Scientific Schools in Russia (project no. NSh-1726.2014.2), and the Government of the Russian Federation (project no. 14.Z50.31.0033 of 24.03.2014).

REFERENCES

1. B. P. Bezruchko, V. I. Ponomarenko, M. D. Prokhorov, D. A. Smirnov, P. A. Tass, Usp. Fiz. Nauk **51** (3), 304 (2008).
2. I. I. Mokhov and D. A. Smirnov, Geophys. Rev. Lett. **33**, L03708 (2006).
3. L. Cimponeriu, M. Rosenblum, and A. Pikovsky, Phys. Rev. E: **70**, 046213 (2004).
4. W. Horbelt, J. Timmer, and H. U. Voss, Phys. Lett. A **299**, 513521 (2002).
5. V. I. Ponomarenko, M. D. Prokhorov, and V. S. Khorev, Tech. Phys. Lett. **39** (8), 683 (2013).
6. D. A. Smirnov, E. V. Sidak, and B. P. Bezruchko, Tech. Phys. Lett. **37** (1), 30 (2011).
7. A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001; Moscow, Tekhnosfera, 2003).
8. D. A. Smirnov, E. V. Sidak, and B. P. Bezruchko, Eur. Phys. J. Spec. Topics **222**, 2441 (2013).
9. A. S. Pikovsky, M. G. Rosenblum, and J. Kurths, Int. J. Bifurc. Chaos **10**, 2291 (2000).
10. Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Berlin: Springer- Verlag, 1984).

Translated by P. Pozdeev