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A Method for Revealing Coupling between Oscillators with Analytical Assessment of Statistical Significance

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Abstract—A method based on calculating the coefficient of correlation between the increments of oscillation phases is proposed for revealing a coupling between two oscillatory systems according to their time series. A distribution of the estimate of this characteristic for uncoupling systems is found; it was used to obtain a criterion for judging the availability of the coupling with a specified confidence probability. The proposed method is simpler than known methods and has a wider range of application, since it also includes oscillators with fairly strong phase nonlinearity. The efficiency of this method is illustrated by examples of reference systems in a numerical experiment.

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A problem of revealing a coupling between two oscillatory systems by time series of their oscillations, i.e., a discrete sequence of values of the observed characteristics is considered in radio physics [1, 2], information transfer [3], biomedical applications [4–6], climatology [7], and other fields. For nonlinear oscillatory systems, approaches based on the introduction and analysis of the oscillation phases $\phi_1(t)$ and $\phi_2(t)$ turn out to be the most sensitive to a weak coupling and, hence, practically efficient [1, 4, 5, 8, 9]. Various phase synchronization indices are used [9]; among them, phase coherence coefficient ρ is the best known. This coefficient is the amplitude of the first Fourier mode of the stationary phase difference distribution [4]: $\rho = \left|\langle e^{i(\phi_1(t) - \phi_2(t))}\rangle\right|$; here and hereinafter, angle brackets mean mathematical expectation. The value of ρ is equal to unity in the case of strict phase synchronization $\phi_1(t) = \phi_2(t)$ and to zero in the case of uncoupled oscillators (without phase nonlinearity) [10]. On this basis, the availability of a coupling is frequently revealed from a significantly nonzero estimate of ρ and statistical significance is checked using surrogate data [11] or special analytical formulas [12]. However, in both cases, it is assumed that the oscillators do not possess individual phase nonlinearity and phase noises are white, which restricts the practical application of this method.

In this work, an alternative approach and the other characteristic of coupling r, i.e., the coefficient of correlation between the phase increments are proposed. A law of the distribution of an estimate of this value for uncoupling systems with almost arbitrary properties of individual phase dynamics is analytically derived. On the basis of this law, a formula for the confidence probability of the difference of this estimate from zero is derived. The efficiency of this approach is shown using examples of reference oscillators with various coupling types and with phase nonlinearity. Conditions for the superiority of this approach over the estimate of phase coherence coefficient ρ in sensitivity are demonstrated.

Let us consider the time series of the oscillation phases of two systems $\{\phi_1(t_1), ..., \phi_1(t_N)\}$ and $\{\phi_2(t_1), ..., \phi_2(t_N)\}$, where $t_n = n\Delta t$ and Δt is the sampling interval. We do not consider methods for calculating the phase [1, 9] and believe that it is correctly obtained for each of the systems under study, for example, by introducing an analytical signal. The empirical estimate of the above-mentioned characteristic ρ is as follows:

$$\hat{\rho} = \left|\frac{1}{N}\sum_{n=1}^{N} e^{i(\phi_1(t_n) - \phi_2(t_n))}\right|,$$

here and hereinafter, the up-arrow means an estimate obtained by a finite length time series.

Let us designate the phase increments during time interval τ as $\Delta\phi_k(t_n) = \phi_k(t_n + \tau) - \phi_k(t_n)$; $k = 1, 2$; $n = 1, ..., N^* = N - \tau/\Delta t$. The coefficient of correlation between the phase increments

$$r = \frac{\langle(\Delta\phi_1 - w_1)(\Delta\phi_2 - w_2)\rangle}{\sigma_{\Delta\phi_1}\sigma_{\Delta\phi_2}}$$

is used as the characteristic of a coupling between the systems, where $w_{1,2} = \langle\Delta\phi_{1,2}\rangle$ are the mathematical expectations of the phase increments; $\sigma_{\Delta\phi_1}$ and $\sigma_{\Delta\phi_2}$ are their standard deviations. For independent from each other systems $r = 0$. If a coupling exists, r can take nonzero values up to unity. Let us use the sample correlation coefficient

$$\hat{r} = \frac{\frac{1}{N}\sum_{i=1}^{N^*}(\Delta\phi_1(t_i) - \hat{w}_1)(\Delta\phi_2(t_i) - \hat{w}_2)}{\hat{\sigma}_{\Delta\phi_1}\hat{\sigma}_{\Delta\phi_2}} \quad (1)$$

as an estimate of characteristic r; here, $\hat{w}_1$ and $\hat{w}_2$ are the sample means and $\sigma_{\Delta\phi_1}$ and $\sigma_{\Delta\phi_2}$ are the sample standard deviations.

To reliably reveal a coupling, it is necessary to check if the estimate $\hat{r}$ significantly differs from zero; this requires a law of its distribution in the case of a zero coupling to be known. Let us derive this law as follows. First, if the series is fairly long, i.e., $N^*\Delta t$ is much more than the autocorrelation times τ_{corr} of the processes $\Delta\phi_1(t)$ and $\Delta\phi_2(t)$, the estimate $\hat{r}$ is the sum of a large number $N^*\Delta t/\tau_{corr}$ of independent terms and, in accordance with the central limit theorem, has a close to Gaussian distribution [13]. Second, sample moment $\hat{r}$ is an asymptotically unbiased estimate [13, 14], so that, for a long series, the mathematical expectation $\hat{r}$ is equal, with a high accuracy, to $\hat{r}$ and, for a zero coupling, to zero. Provided that the phase increments have a normal or close to normal distribution, the dispersion of $\hat{r}$ is described by Bartlett's formula [14] that takes the following form for uncoupling oscillators:

$$\sigma_{\hat{r}}^2 = \frac{1}{N^*}\sum_{n=-\infty}^{\infty} c_{\Delta\phi_1}(n\Delta t)c_{\Delta\phi_2}(n\Delta t),$$

where $c_{\Delta\phi_1}(n\Delta t)$ and $c_{\Delta\phi_2}(n\Delta t)$ are the autocorrelation functions of $\Delta\phi_1(t)$ and $\Delta\phi_2(t)$. We obtain an estimate of the dispersion through the sample estimates $c_{\Delta\phi_1}$ and $c_{\Delta\phi_2}$ as follows:

$$\hat{\sigma}_{\hat{r}}^2 = \frac{1}{N^*}\sum_{n=-N^*/4}^{N^*/4} \hat{c}_{\Delta\phi_1}(n\Delta t)\hat{c}_{\Delta\phi_2}(n\Delta t).$$

Thus, the 95% confidence interval for the characteristic r is $\hat{r} \pm 1.96\hat{\sigma}_{\hat{r}}$ and, during an analysis of the time series, a conclusion is drawn on the availability of the coupling, i.e., a positive conclusion is drawn when $|\hat{r}| > 1.96\hat{\sigma}_{\hat{r}}$ with a confidence probability of 0.95, i.e., at a significant level of 0.05. Below, we present an analysis of the efficiency of this approach that includes the estimates of probabilities of positive conclusions without a coupling (these are false conclusions the probability of which should not exceed 0.05) and with a coupling available (these are true conclusions the probability of which governs the sensitivity of the method).

We use the following phase oscillators as reference systems:

$$\begin{aligned} d\phi_1/dt &= \omega_1 + b\sin\phi_1 + k_{d,1}\sin(\phi_2 - \phi_1) \\ &\quad + k_m\sin\phi_2 + \xi_1(t), \\ d\phi_2/dt &= \omega_2 + b\sin\phi_2 + k_{d,2}\sin(\phi_1 - \phi_2) \\ &\quad + k_m\sin\phi_1 + \xi_2(t), \end{aligned} \quad (2)$$

where ω_1 and ω_2 are the angular frequencies; b is the phase nonlinearity parameter; $k_{d,1}$ and $k_{d,2}$ are the "difference" coupling coefficients; k_m is the modulating coupling coefficient; the noises ξ_1 and ξ_2 are independent and have the autocorrelation functions $\langle\xi_k(t)\xi_k(t')\rangle = \sigma_{\xi_k}^2\delta(t - t')$; $k = 1, 2$; δ is the delta function; and $\sigma_{\xi_1}^2$ and $\sigma_{\xi_2}^2$ are the intensities of the noises (see, e.g., [1, 2, 9, 10]). We consider either only a difference ($k_m = 0$) or only a modulating ($k_{d,1} = k_{d,2} = 0$) coupling. The difference coupling is considered in the symmetric ($k_{d,1} = k_{d,2} = k_d$) and "antisymmetric" ($k_{d,1} = -k_{d,2} = k_d$) forms; in both cases, $b = 0$. In the first case, the coupling is synchronizing (for zero noise and a low frequency mismatch, the phase synchronization mode 1 : 1 becomes stable at $k_d > |\omega_1 - \omega_2|/2$) and, in the second case, the coupling is nonsynchronizing. The modulating coupling is considered for "linear" ($b = 0$) and "nonlinear" ($b \neq 0$) oscillators.

For each set of the parameters, we analyzed an ensemble of $M = 100$ pairs of time series obtained by integrating Eqs. (2) with a step of 0.01 using the Euler method. The sampling interval was 0.3, i.e., 20 points during a characteristic perimeter; the series length was $N = 2000$ or about 100 characteristic periods. The results presented below were obtained for a value of τ equal to two characteristic periods, but they are similar at any value of τ that exceeds about one-fourth of the characteristic period. For each pair of the series, the value of $\hat{r}$ was calculated and a conclusion on the availability of a coupling was drawn or not drawn by the criterion $|\hat{r}| > 1.96\hat{\sigma}_{\hat{r}}$. We calculated the frequency or the estimate of the probability of positive conclusions f, i.e., a fraction of the time series for which a conclusion on the availability of a coupling was drawn.

Figure 1 illustrates values of f and ensemble averages $\langle\hat{\tau}\rangle$ for the set of the parameters $\omega_1 = 1.1$, $\omega_2 = 0.9$, $\sigma_{\xi_1} = 0.2$, and $\sigma_{\xi_2} = 0.1$, i.e., for nonidentical oscillators with a moderate noise level. They show that the proposed method is correct since the frequency of false conclusions does not exceed 0.05 (see the dashed lines at $k_d = 0$ or $k_m = 0$). In addition, this method is fairly sensitive to a difference synchronizing coupling, i.e., frequency f is high even at low, as compared with the value $(\omega_1 - \omega_2)/2 = 0.1$, values of k_d and grows with increasing k_d (Fig. 1a). For an antisymmetric differ-

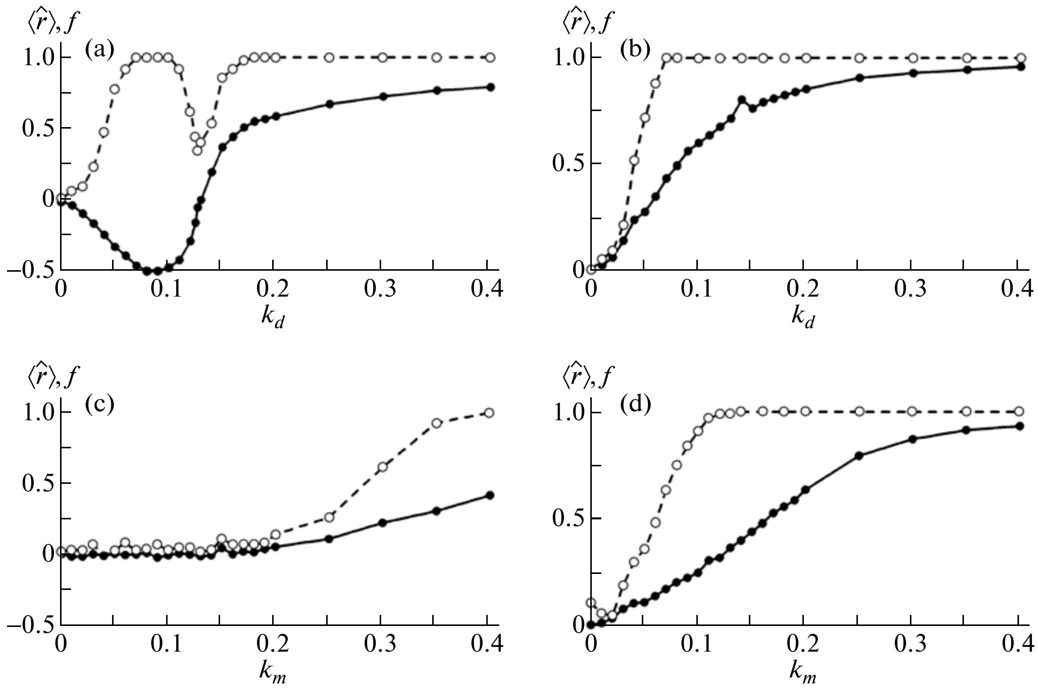

Fig. 1. Average values of $\langle \hat{r} \rangle$ (solid lines) and frequencies of positive conclusions f (dashed lines) calculated for ensemble of 100 time series: (a) difference coupling with $k_{d,1} = k_{d,2} = k_d$, (b) difference coupling with $k_{d,1} = -k_{d,2} = k_d$, and (c, d) modulating coupling at (c) $b = 0$, (d) $b = 0.7$.

ence coupling, f increases still more rapidly (Fig. 1b). Sensitivity to a modulating coupling is somewhat lower, but also fairly high (Figs. 1c and 1d), especially when phase nonlinearity exists (Fig. 1d).

Somewhat unexpected negative correlations of r and the nonmonotonic pattern of the graphs presented in Fig. 1a for the synchronizing difference coupling are explained as follows. We note that the right side of the first and second equations of system (2) contains the same term $k\sin(\phi_2 - \phi_1)$, but the signs of this term differs for these equations. The phase increments are obtained by integrating system (2) over interval τ, so that

$$\eta(t) = \int_t^{t+\tau} k\sin(\phi_2(t') - \phi_1(t'))dt'$$

is a component of both phase increments. At $b = 0$, we have $\Delta\phi_1(t) = \omega_1\tau + \varepsilon_1(t) + \eta(t)$ and $\Delta\phi_2(t) = \omega_2\tau + \varepsilon_2(t) - \eta(t)$, where $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are integrals of $\xi_1(t)$ and $\xi_2(t)$; i.e., independent Gaussian processes with zero average value. In the case of a weak coupling, $\varepsilon_k(t)$ and $\eta(t)$ can be assumed to be independent of another. Since common additive component $\eta(t)$ with different signs is available, we obtain a negative correlation of r (Fig. 1a, weak couplings). In the mode close to synchronization, the mutual independence of $\varepsilon_k(t)$ and $\eta(t)$ is violated and the phase increments during one and the same time interval are almost equal, so that the value of r is positive and close to unity (Fig. 1a, strong couplings). Therefore, r changes its sign from minus to plus as the mode approaches the synchronization mode and f has a dip at intermediate values of k_d when the graph of r intersects the straight line $r = 0$.

In the case of the antisymmetric difference coupling, the total component of the phase differences $\eta(t)$ has one and the same sign and results in a positive correlation and a monotonic increase in r and f (Fig. 1b). A similar analysis carried out for the modulating coupling shows that the value of r is sensitive to this coupling in the case of phase nonlinearity $b \neq 0$ when the right sides of the first and second equations of system (2) contain common terms. This is in fact observed in Fig. 1d. However, it is interesting that this method is also sensitive to the modulating coupling in the case of $b = 0$, but only at sufficiently higher values of k_m (Fig. 1c). An analogous analysis shows that, in the case of a unidirectional coupling, all results are similar, but the sensitivity of the method somewhat decreases (the graphs are not shown); the method is totally insensitive only in the case of a unidirectional modulating coupling.

Figure 2 shows that the use of r provides some advantages over the use of ρ in addition to greater simplicity and versatility during the assessment of significance. Namely, the value of r is sensitive to the modulating coupling (Figs. 2c and 2d), where ρ is totally insensitive since the phase difference distribution

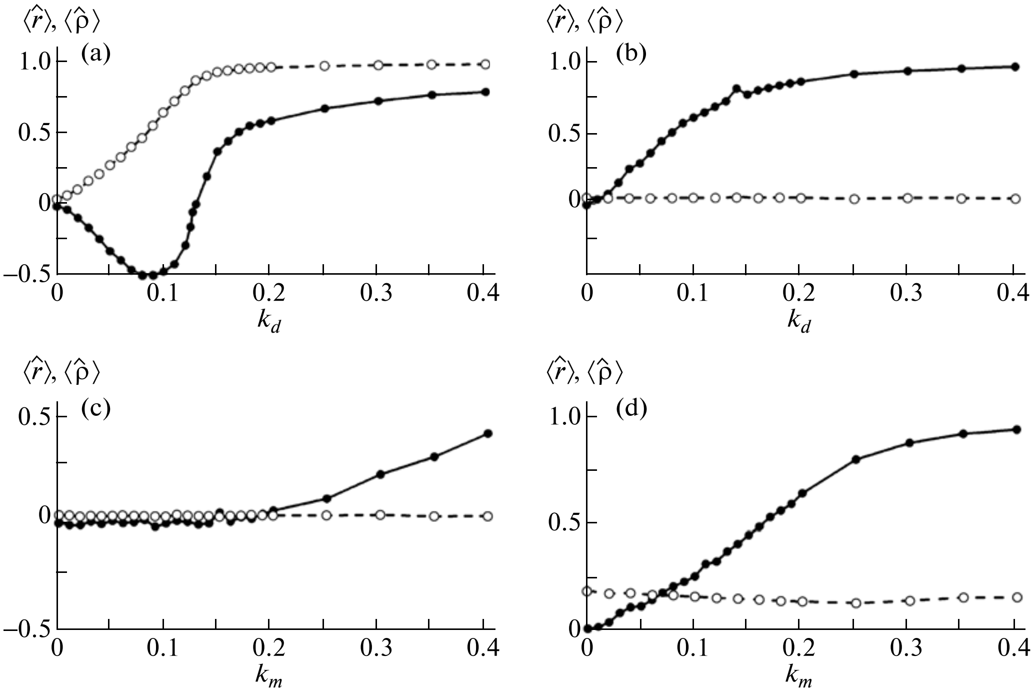

Fig. 2. Dependences of phase coherence coefficients (dashed lines) and coefficients of correlation between phase increments (solid lines) on coupling coefficients: (a) synchronizing coupling, (b) antisymmetric difference coupling, and (c, d) modulating coupling at (c) $b = 0$ and (d) $b = 0.7$.

remains almost uniform with increasing intensity of the coupling. A similar situation arises in the case of the antisymmetric difference coupling (Fig. 2b). For the synchronizing difference coupling, the absolute values of r and ρ grow with increasing k at an almost identical rate (Fig. 2a). A similar analysis shows that ρ has greater advantages in sensitivity only in the case of the unidirectional synchronizing coupling (graphs are not shown).

We emphasize that the conclusion on the existence of a coupling drawn on the basis of a nonzero value of the estimate ρ with a specified confidence probability is only possible for oscillators without phase nonlinearity [10]. Otherwise, a special analysis is needed; for example, in Fig. 2d, ρ is high even for the uncoupling oscillators and significantly differs from zero at $b \neq 0$. This is due to phase nonlinearity that leads to the nonuniformity of the phase difference distribution within the section from 0 to 2π rather than to the existence of the coupling. The alternative approach proposed in this work is also efficient for nonlinear oscillators without any changes.

In this work, a method for revealing a coupling between oscillators is proposed that makes it possible to draw conclusions with a specified confidence probability. It is based on calculating correlations between the increments of phases of oscillations. This method is easy to implement and does not require a large computation volume, unlike known approaches that involve the construction of surrogate data [11]. Unlike the widely used estimate of the phase coherence coefficient, it is applicable for a broad range of cases for various coupling types and with individual phase nonlinearity in the dynamics of the systems under study [12]. Therefore, this method seems to be a useful new means for investigating the coupling between oscillatory systems of various natures according to their time series.

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