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Detection of coupling between oscillators with analytic tests for significance

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Abstract. To detect coupling between two oscillators from a time series, we suggest a method based on the estimation of the phase increments correlation with an analytic test for significance. With exemplary oscillators, we show that the suggested method complements a widely used approach based on the estimation of the mean phase coherence. In particular, the suggested method allows efficient detection of a non-synchronizing coupling and, due to a less restrictive null hypothesis, it is applicable to a wider range of situations, including arbitrarily strong phase nonlinearities.

1 Introduction

In studies of coupled oscillatory systems, much attention has been paid to the phenomenon of chaotic synchronization (e.g. [1–6]), including the "inverse" problems of detection and characterization of synchronous regimes from time series (e.g. [7–12]). Many important contributions to this field have been made by Prof. V.S. Anishchenko and his collaborators, see e.g. Refs. [2,3,6,12] and multiple references therein. This work is devoted to a similar problem of detecting the very presence of coupling from a time series, which is also widespread in different fields such as cardiology [10,13], neurophysiology [7,9,14], and climatology [15,16].

The existing coupling detection techniques [17,18] with analytic tests for significance (which are highly desirable since any bootstrap, e.g. [17], or surrogate data, e.g. [19], tests require an essential computation time and quite enough observation data) are based on the estimation of the mean phase coherence (MPC) [7], which is the first Fourier mode of the phase difference distribution [8] and is widely used as a phase synchronization index as well [1]. In Ref. [17], the authors managed to develop an analytic test against the null hypothesis of uncoupled oscillators under an additional assumption that the observed phase difference values are statistically independent of each other. The latter strongly restricts applicability of the technique in practice, since observed phases typically exhibit nonzero autocorrelations. Ref. [18] represents a step forward by avoiding this restriction: an additional assumption used

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is that phase dynamics of the systems under study are governed by the equations of the first-order phase oscillators with Gaussian white noise and without individual phase nonlinearities. It allows to account for the phase autocorrelations and promises efficiency of the approach for a variety of practical situations.

However, there are quite usual circumstances where properties of the systems differ from the above assumption that can lead to spurious inferences of couplings. Besides, an analysis of the phase difference implies an opportunity to detect only such couplings which enhance synchrony between oscillators. Yet, some kinds of coupling do not synchronize the oscillations but essentially influence the dynamics in other ways, e.g. by frequency modulation. To overcome these difficulties, we suggest here a complementary method based on the estimation of the phase increments correlation (PIC). We derive a simple formula to test against a null hypothesis of uncoupled oscillators under a rather general additional condition of stationary phase increments. As we show below, the PIC method can detect non-synchronizing couplings and appears applicable in some situations where the MPC method [18] fails. In a brief version, the PIC method has been reported in the Letter [20], while here we discuss it in detail and compare with the MPC method thoroughly. Below, Sect. 2 describes the MPC and PIC methods, Sect. 3 compares results of their applications to exemplary oscillators, while a discussion and conclusions are given in Sect. 4.

2 Coupling detection methods

To introduce necessary notations, let us repeat the problem formulation. There are two oscillatory systems, and no third system influences both of them. The task is to find out whether the two systems are coupled on the basis of a time series of their phases: $\phi_1(t_1), \phi_1(t_2), \ldots, \phi_1(t_N)$ and $\phi_2(t_1), \phi_2(t_2), \ldots, \phi_2(t_N)$ where $t_n = n\Delta t$, Δt is a sampling interval, and N is a time series length. These are "unwrapped" phases, i.e. increasing by 2π after each full cycle. Speaking mathematically, the task is to find out whether the null hypothesis of statistically independent random processes $\phi_1(t)$ and $\phi_2(t)$ (i.e. any of their finite-dimensional joint distributions is a product of the respective individual distributions) agrees with the observed data. In statistical terms, one checks whether this hypothesis can be rejected at a given sufficiently small significance level p (the probability of a random error which is often fixed at 0.05). The above problem formulation is rather general since no other properties of the processes are specified. It is necessary to make the null hypothesis more concrete in order to develop a practical test against it as discussed in Sects. 2.1 and 2.2.

2.1 Mean phase coherence-based method

Following Ref. [18], presume that the observed phase dynamics are governed by the equations of uncoupled first-order phase oscillators without phase nonlinearities:

$$d\phi_k(t)/dt = \omega_k + \xi_k(t), \quad k = 1, 2, \tag{1}$$

where ω_k are angular frequencies, ξ_k are Gaussian white noises independent of each other, their autocorrelation functions are $\langle \xi_k(t)\xi_k(t')\rangle = \sigma^2_{\xi_k}\delta(t-t')$ with $\sigma^2_{\xi_k}$ representing the noise intensities. This is a particular case of the general null hypothesis of independent processes. Note that such equations describe self-sustained oscillations in many real systems to a good accuracy (e.g. [21], [22]), so that the null hypothesis seems concrete but not too restrictive.

Denote the "wrapped" phase difference $\Phi(t) = (\phi_1(t) - \phi_2(t)) \bmod 2\pi$, which is a stationary process for the system (1). MPC is defined as the amplitude of the first

Fourier mode of the one-dimensional probability density of Φ: $\rho = |\langle \exp(i\Phi)\rangle|$, angular brackets stand for the expectation value throughout the paper. $\rho = 1$ in the case of strict synchrony $\phi_1(t) = \phi_2(t) + const$, and $\rho = 0$ for uncoupled oscillators with uniform distribution of Φ that holds true for Eqs. (1).

As a statistic distinguishing between the null hypothesis (1) and coupled oscillators, the MPC estimator $\hat{\rho} = |(1/N)\sum_{k=1}^{N} \exp(i\Phi(t_k))|$ appears useful, a "hat" denotes an estimate from a time series throughout the paper. For uncoupled oscillators, the $(1-p)$-quantile of the distribution of $\hat{\rho}$ is approximated asymptotically as $C\chi^2_{1,1-p}$, where $\chi^2_{1,1-p}$ is the $(1-p)$-quantile of the χ^2 distribution with one degree of freedom and $C = \frac{1}{N}\left(1 + 2\frac{e^{-D\Delta t/2}\cos(\Delta\omega\Delta t) + e^{-DN\Delta t/2}\cos(\Delta\omega N\Delta t) - e^{-D\Delta t}}{1 - 2e^{-D\Delta t/2}\cos(\Delta\omega\Delta t) + e^{-D\Delta t}}\right)$ with $\Delta\omega = \omega_1 - \omega_2$ and $D = \sigma^2_{\xi_1} + \sigma^2_{\xi_2}$ [18]. $\Delta\omega$ and D can be estimated from a time series as $\Delta\hat{\omega} = \frac{\sum_{n=1}^{N} t_n \Phi(t_n)}{\sum_{n=1}^{N} t_n^2}$ and $\hat{D} = \frac{\sum_{n=1}^{K}(\Phi(t_{nl}) - \Phi(t_{(n-1)l}) - l\Delta\hat{\omega}\Delta t)^2}{lK\Delta t}$, where $N = lK$, that gives an estimate $\hat{C}$. The null hypothesis is rejected (i.e. coupling is detected, if all other conditions are unchanged) at $p = 0.05$, if $\hat{\rho} > \hat{C}\chi^2_{1,0.95}$ [18].

2.2 Phase increments correlation-based method

Denote with $\Delta\phi_k(t) = \phi_k(t+\tau) - \phi_k(t)$, $k = 1, 2$, the phase increments over a certain time interval τ. Presume that they are stationary random processes whose autocorrelations decay fast enough (over time much less than a time series length). As it follows from the general hypothesis of uncoupled systems, $\Delta\phi_1(t)$ and $\Delta\phi_2(t)$ are also statistically independent of each other. We suggest to check the latter property whose rejection would imply the presence of coupling (for all other conditions unchanged). Denote expectations of the phase increments $w_k = \langle \Delta\phi_k \rangle$ and their variances $\sigma^2_{\Delta\phi_k}$. The phase increments correlation $r = \frac{\langle(\Delta\phi_1(t) - w_1)(\Delta\phi_2(t) - w_2)\rangle}{\sigma_{\Delta\phi_1}\sigma_{\Delta\phi_2}}$ is correctly defined if the joint distribution of $\Delta\phi_1(t)$ and $\Delta\phi_2(t)$ possesses finite moments of the first and the second order. For statistically independent $\Delta\phi_1$ and $\Delta\phi_2$, one gets $r = 0$. In the case of nonzero coupling, r can differ from 0 and reach 1 in absolute value.

Denote $N^* = N - \tau/\Delta t$ the length of the time series $\Delta\phi_k(t_n), n = 1, \ldots, N^*$. As a statistic used to reject the null hypothesis, we suggest the sample estimate

$$\hat{r} = \frac{(1/N^*)\sum_{n=1}^{N^*}(\Delta\phi_1(t_n) - \hat{w}_1)(\Delta\phi_2(t_n) - \hat{w}_2)}{\hat{\sigma}_{\Delta\phi_1}\hat{\sigma}_{\Delta\phi_2}}, \quad (2)$$

where $\hat{w}_k$ are sample means and $\hat{\sigma}^2_{\Delta\phi_k}$ sample variances. Under the null hypothesis, the distribution of $\hat{r}$ is asymptotically Gaussian with zero mean [23]. In the case of Gaussian distributions of the phase increments, its variance $\sigma^2_{\hat{r}}$ can be computed via the Bartlett formula [24]: $\sigma^2_{\hat{r}} = (1/N^*)\sum_{n=-\infty}^{\infty} c_{\Delta\phi_1}(n\Delta t)c_{\Delta\phi_2}(n\Delta t)$, where $c_{\Delta\phi_k}(n\Delta t)$ are the autocorrelation functions of $\Delta\phi_k(t)$. An estimate $\hat{\sigma}^2_{\hat{r}}$ can be obtained from the sample autocorrelations as $\hat{\sigma}^2_{\hat{r}} = (1/N^*)\sum_{n=-N^*/4}^{N^*/4} \hat{c}_{\Delta\phi_1}(n\Delta t)\hat{c}_{\Delta\phi_2}(n\Delta t)$. Alternatively, the summation can be performed only up to the autocorrelation times of $\Delta\phi_k$ that gives the same results for reasonably quickly decaying autocorrelations. Thus, the null hypothesis is rejected at $p = 0.05$, if $|\hat{r}| > 1.96\hat{\sigma}_{\hat{r}}$.

3 Applications and comparison of the methods

The MPC and PIC methods are applied to simulated time series of exemplary systems with a "synchronizing" coupling (Sect. 3.1) and a "non-synchronizing" one (Sect. 3.2).

Systems with non-Gaussian and non-white phase noises are considered in Sect. 3.3 to illustrate generality of the results.

As benchmark systems in Sect. 3.1 and 3.2, we use the phase oscillators

$$\begin{aligned} d\phi_1(t)/dt &= \omega_1 + k_1 \sin(\phi_2(t) - \phi_1(t)) + \xi_1(t), \\ d\phi_2(t)/dt &= \omega_2 + k_2 \sin(\phi_1(t) - \phi_2(t)) + \xi_2(t), \end{aligned} \tag{3}$$

where k_1, k_2 are coefficients of the difference coupling which is considered in the form of bidirectional symmetric ($k_1 = k_2 = k$, it is a "synchronizing" coupling since in the noise-free case a strictly synchronous regime 1:1 becomes stable at $k > k_c = |\Delta\omega|/2$), bidirectional antisymmetric ($k_1 = -k_2 = k$, it is a "non-synchronizing" coupling since synchronization does not occur at any k), or unidirectional ($k_1 = 0, k_2 = k$, a synchronizing coupling).

In Sect. 3.2, the phase oscillators with a "modulating" coupling are also used:

$$\begin{aligned} d\phi_1(t)/dt &= \omega_1 + b \sin\phi_1(t) + k_1 \sin\phi_2(t) + \xi_1(t), \\ d\phi_2(t)/dt &= \omega_2 + b \sin\phi_2(t) + k_2 \sin\phi_1(t) + \xi_2(t), \end{aligned} \tag{4}$$

where b is a coefficient of individual phase nonlinearity, and coupling is considered as bidirectional symmetric ($k_1 = k_2 = k$) or unidirectional ($k_1 = 0, k_2 = k$). This is a kind of non-synchronizing coupling.

Section 3.3 considers the Roesller systems with synchronizing coupling as in Ref. [18]:

$$\begin{aligned} \frac{dx_1}{dt} &= -\omega_1 y_1 - z_1 + k(x_2 - x_1) + \xi_1, \ \frac{dy_1}{dt} = \omega_1 x_1 + a y_1, \ \frac{dz_1}{dt} = b + (x_1 - c) z_1, \\ \frac{dx_2}{dt} &= -\omega_2 y_2 - z_2 + k(x_1 - x_2) + \xi_2, \ \frac{dy_2}{dt} = \omega_2 x_2 + a y_2, \ \frac{dz_2}{dt} = b + (x_2 - c) z_2. \end{aligned} \tag{5}$$

In addition, a "modulating" coupling is checked as described in Sect. 3.3.

Since Ito and Stratonowich stochastic integrals coincide in the considered case of constant noise intensities, the standard Euler technique is used for numerical integration in all examples. For each benchmark system, we vary coupling strength k in a certain range and at each value of k generate an ensemble of 100 time series using the integration step 0.01, sampling interval $\Delta t = 0.3$ (about 21 points per a mean oscillation period), and the length $N = 2094$ (about 100 mean periods), if not otherwise stated. From each time series, ρ and r are estimated and a positive (i.e. a rejection of the null hypothesis) or negative conclusion about coupling existence is made. The frequency f of positive conclusions is computed for each of the two methods. For uncoupled systems, f must not exceed the claimed level of $p = 0.05$ within statistical fluctuations related to the finite ensemble size (the values of f up to 0.1 are allowable in the considered setting of 100 time series in an ensemble) in order that a method can be regarded applicable. If this is the case for zero coupling, then for non-zero couplings f characterizes the sensitivity: the greater f, the better the method.

3.1 Synchronizing coupling

Figure 1 shows estimation results for the system (3) with a bidirectional symmetric coupling, a moderate frequency mismatch $\omega_1 = 1.1$, $\omega_2 = 0.9$, and moderate noise levels $\sigma_{\xi_1} = 0.2$, $\sigma_{\xi_2} = 0.1$. The PIC method is applied with $\tau = 40\Delta t$ (about two basic periods), the choice being discussed below. Note that it works properly, i.e. does not give large false positive rates (the solid line at $k = 0$ in Fig. 1b). It is

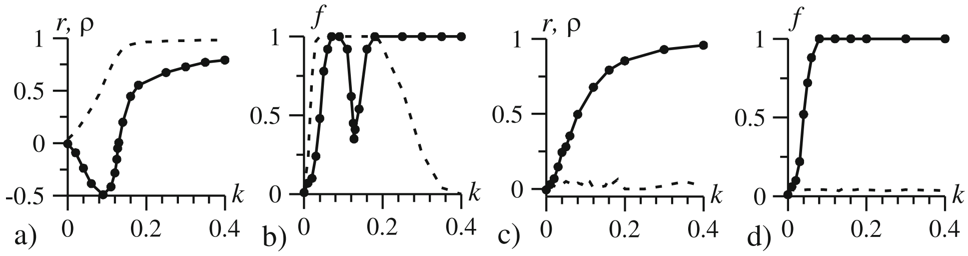

Fig. 1. Performance of the MPC (the dashed lines) and PIC (the solid lines) methods for the system (3) with bidirectional symmetric (a,b) or antisymmetric (c,d) couplings: (a,c) r stands for $\hat{r}$ averaged over an ensemble of 100 time series (the solid line) and similarly for ρ (the dashed line); (b,d) the respective rates of positives f.

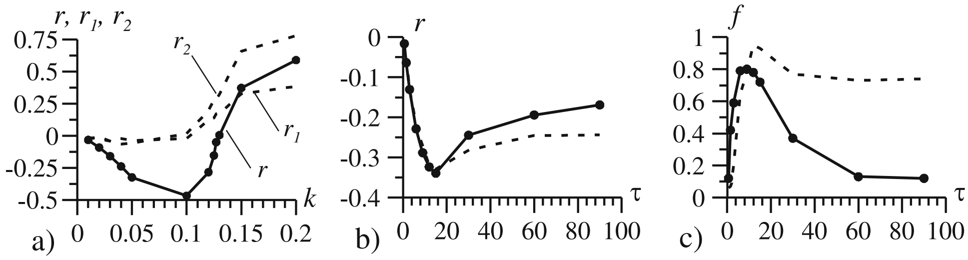

Fig. 2. Details of the PIC method performance for the system (3) with a bidirectional symmetric coupling: (a) the value of r (the solid line) along with correlations between η and ϵ_1 (r_1, a dashed line), η and ϵ_2 (r_2, a dashed line); (b,c) r and f versus τ for $k = 0.05$ and $N = 2094$ (i.e. about 100 basic periods, the solid lines) or $N\Delta t = 100\tau$ (the dashed lines).

reasonably sensitive: f is large already for small enough couplings (e.g. $f = 0.77$ at $k = 0.05 < k_c = 0.1$) and rises with k. All this concerns the MPC method too, which appears even more sensitive to weak coulings (Fig. 1b, the dashed line).

Note the non-monotone behavior of r versus k (Fig. 1a) and negative r's at small k that can be explained as follows. Both equations in (3) include the same term $k \sin(\phi_2(t) - \phi_1(t))$ with the opposite signs. The phase increments are integrals of the left-hand sides of Eqs. (3) over an interval τ, hence, $\eta(t) = \int_t^{t+\tau} \sin(\phi_2(t') - \phi_1(t'))dt'$ is a component of both phase increments with the opposite signs: $\Delta\phi_1(t) = \omega_1\tau + k\eta(t) + \epsilon_1(t)$ and $\Delta\phi_2(t) = \omega_2\tau - k\eta(t) + \epsilon_2(t)$, where ϵ_1, ϵ_2 are integrals of ξ_1, ξ_2, i.e. zero-mean Gaussian white noises independent of each other. At weak couplings, ϵ_k and η are almost uncorrelated: indeed, a large value of ϵ_k induces a large value of the argument of the sine function in η, but the number of its oscillations over an interval τ is related to ϵ_k in a complicated way, leading to small correlation between the resulting η and ϵ_k. This consideration is confirmed by the estimation of these terms from a simulated time series (Fig. 2a). Thus, one gets $r \approx \frac{-k^2\langle\eta^2\rangle}{\sigma_{\xi_1}\sigma_{\xi_2}\tau}$. At large k, close to a synchronous regime, the two simultaneous phase increments are almost equal to each other so that r is close to unity (Fig. 1a). Thus, r changes its sign from minus to plus with rising k. This is accompanied by appearance of nonzero correlations between η and ϵ_k (Fig. 2a).

The PIC and MPC methods differ in their sensitivity to weak couplings because they characterize different properties of the phase dynamics. MPC "extracts" just the first Fourier mode of the phase difference Φ and ignores everything else. This mode changes first when a synchronizing coupling strength rises. Therefore, MPC is probably the most sensitive characteristic to such coupling. Rather, PIC reflects the presence of common terms in the right-hand sides of the phase dynamics equations of both oscillators, i.e. it characterizes "driving forces" of the phase dynamics. However, PIC is influenced by both the common terms and all other terms, in particular, by the phase noises ξ_k, that decreases its value and, hence, the sensitivity of the method. But the feature that PIC does not rely only on the phase difference distribution makes it much better than MPC for detection of non-synchronizing couplings (see below).

As for the choice of τ, the values of r and f weakly depend on τ in the range from one to four basic periods at the time series length of 100 basic periods (Figs. 2b,c, the solid lines). τ equal to two basic periods seems to give a bit higher f and is used throughout the paper (since similar dependencies of f on τ are observed in all examples). Too small τ's (less than a period, where a phase increment is proportional to an "instantaneous" oscillation frequency) and too big ones (greater than five periods) lead to considerably lower f. For longer time series, an optimal τ can be shifted to somewhat greater values (Fig. 2c, the dashed lines). Thus, in practice one can use the entire plot of $\hat{r}$ versus τ for a more complete characterization of interdependencies.

Already in this example, we can mention two special cases when the PIC method is advatangeous. First, strong couplings lead to a decrease in MPC sensitivity down to zero (Fig. 1b), i.e. it cannot distinguish between the situations of synchronized oscillators and almost identical uncoupled ones. The reason is that the estimate $\Delta\hat{\omega}$ approaches zero in a synchronous regime (i.e. becomes too inaccurate) and, hence, leads to an overestimated 0.95-quantile of the $\hat{\rho}$ distribution under the null hypothesis. The PIC method does not encounter any such difficulty because it does not require accurate estimation of parameters in any dynamical equations. Second, the PIC method appears less demanding to the time series length. Namely, both MPC and PIC methods use asymptotic formulas and, hence, can be erroneous for too short time series. Indeed, the MPC starts to give more errors than allowed for time series shorter than 50 basic periods as our calculations evidence (the plots are not shown). In particular, for the system (3) with $k = 0$ the MPC method gives $f = 0.26$ for the time series of a 30 periods length ($N = 628$) and $f = 0.37$ for ten periods ($N = 210$). Surprisingly, the PIC method performs successfully even for such short time series giving the false positive rate $f < 0.05$ as required. This property is maintained under variation of parameters (in particular, noise levels) in a wide range. Seemingly, such a higher robustness of the PIC method is due to its simpler formalism and independence of model parameter estimates, which is a manifestation of its less restrictive null hypothesis. At that, the PIC method maintains a reasonably high sensitivity in the case of short time series, e.g. at $k = 0.1$ it gives $f = 0.5$ for a thirty periods length and $f = 0.18$ for ten periods, i.e. considerable values (and they rise with k).

The case of unidirectional coupling is rather similar. The only difference is that the PIC method is even less sensitive to weak couplings (the plots are not shown).

3.2 Non-synchronizing coupling

For the oscillators (3) with antisymmetric coupling, the PIC method is quite sensitive to weak couplings (Fig. 1d). Moreover, the value of r rises monotonically with k (Fig. 1c) since the common term η now enters both equations in (3) with the same sign leading to a positive PIC. The MPC method is completely insensitive to such coupling because the distribution of the phase difference Φ remains uniform for any k as can be rigorously shown by subtracting the two equations of (3) from each other.

For the oscillators with modulating coupling (4), the sensitivity of the PIC method depends on the strength of the phase nonlinearity b (Fig. 3). Its sensitivity to weak couplings is high enough for a large $b = 0.7$ (Figs. 3c,d), while it is much lower for $b = 0$ (Figs. 3a,b). In the first case, the high sensitivity is due to the common terms

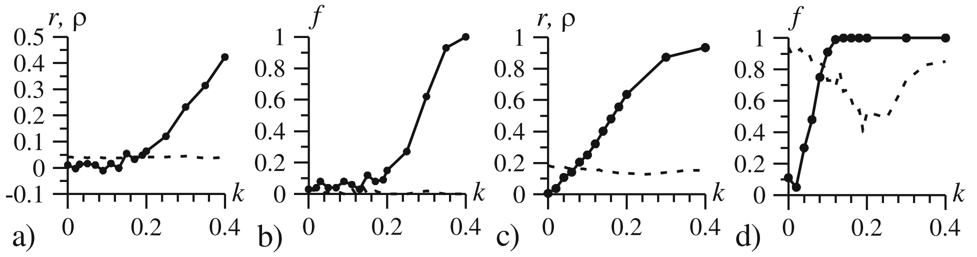

Fig. 3. Performance of the MPC (dashed lines) and PIC (solid lines) methods for the system (4) with bidirectional coupling at $b = 0$ (a,b) and $b = 0.7$ (c,d). As in Fig. 1, r and ρ are the values of sample estimates averaged over ensembles of time series (a,c), f are the respective rates of positives (b,d).

in the right-hand sides of the phase dynamics equations for the two oscillators: phase nonlinearity of each oscillator is described by the term proportional to its influence on the other oscillator. In the second case, only a very strong coupling is detected by the PIC method, seemingly, due to arousal of nonzero correlations between the terms like η and ϵ_k analogously to the discussion in Sect. 3.1.

In this example, the MPC method is completely insensitive even to quite strong coupling for $b = 0$ (Fig. 3b, the dashed line almost coinciding with the abscissa axis). For a strong phase nonlinearity (Figs. 3c,d), the MPC method exhibits nonzero values of ρ even for uncoupled oscillators: indeed, the phase nonlinearity leads to non-uniformity of the phase difference Φ distribution which has some "preferable" (more probable) values since the wrapped phases of both oscillators have their preferable values as well. Thus, a conclusion about coupling presence based on the MPC method can be erroneous for oscillators with a considerable phase nonlinearity. To avoid such errors, one would need a special analysis, e.g. a preliminary transformation to "genuine" phases [25], that needs a further investigation. As for the PIC method, it works well without any change in the case of strong phase nonlinearity since its null hypothesis does not pose any restrictions on this property of the phase dynamics. Qualitatively the same results with minor quantitative variations are obtained for unidirectional coupling $k_1 = 0, k_2 > 0$ (the plots are not shown).

Thus, we have observed the following advantages provided by the PIC method: (i) it is sensitive to non-synchronizing couplings where the MPC method is completely insensitive; (ii) it is applicable in the case of an arbitrarily strong phase nonlinearity, where the MPC method can lead to erroneous coupling detections. Recall that the MPC method implies Gaussian and white phase noises, while the PIC method presumes Gaussian distributions of the phase increments. Both properties holds true under the null hypothesis of the uncoupled oscillators (1). In all examples above, they are met to a good accuracy. Let us check whether these requirements are critical by considering oscillators with rather different properties of the phase noises.

3.3 Non-Gaussian and non-white phase noises

Consider the system (5) with parameters $a = 0.15, b = 0.2, c = 10, \omega_1 = 1.015, \omega_2 = 0.985$ as in Ref. [18], which provide chaotic oscillations in the noise-free case. We obtain the phases from time series of the variables x_1, x_2 using the Hilbert transform (see e.g. [8], [18]). It can be easily verified that properties of the phase noises in this system differ from the "Gaussian and white" condition, because these noises are determined to a significant extent by the slowly varying chaotic amplitudes which are affected by the phases in their turn. This is manifested also in the properties of

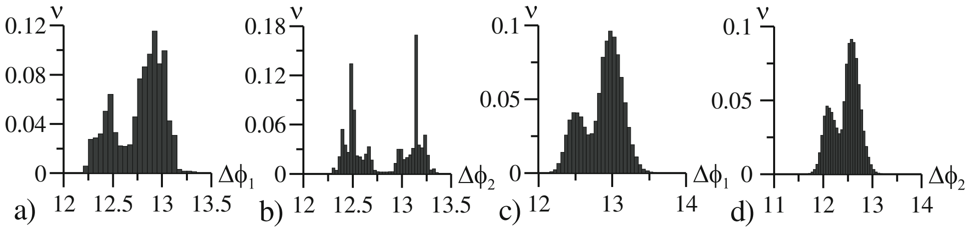

Fig. 4. Histograms of the phase increments over the interval $\tau = 12$ (about two basic periods) for the Roessler systems (5): (a,b) in the noise-free case $\sigma_{\xi_1} = \sigma_{\xi_2} = 0$; (c,d) in the noisy case $\sigma_{\xi_1} = \sigma_{\xi_2} = 0.2$. The panels (a,c) correspond to the first system, (b,d) to the second one. Relative frequencies of falling into a bin are shown along the ordinate axes.

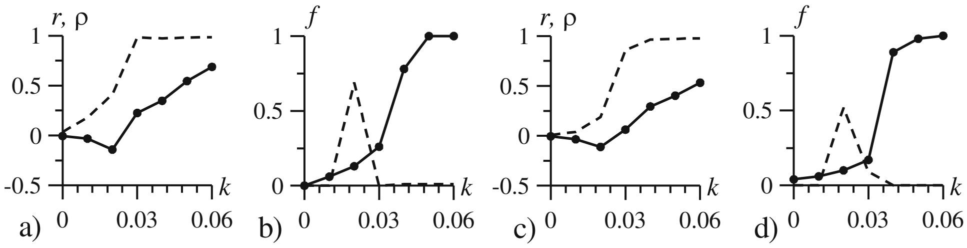

Fig. 5. Performance of the MPC (dashed lines) and PIC (solid lines) methods for the system (5): (a,b) in the noise-free case $\sigma_{\xi_1} = \sigma_{\xi_2} = 0$; (c,d) in the noisy case $\sigma_{\xi_1} = \sigma_{\xi_2} = 0.2$. The panels (a,c) correspond to r and ρ, which are the sample estimates averaged over ensembles of time series, the panels (b,d) to the respective rates of positives f.

the phase increments, whose distributions are shown in Fig. 4. They are essentially non-Gaussian, especially in the noise-free case (Figs. 4a,b).

The performance of the MPC and PIC methods is presented in Fig. 5. The results appear quite similar to those observed for the phase oscillators in Figs. 1a,b. As well, we have obtained very similar results for another canonical set of parameters corresponding to a spiral chaos $a = 0.2, b = 0.2, c = 6.1$ (see e.g. [26]), where the phase increments distributions are again non-Gaussian but exhibit a single maximum (the plots are not shown). An example with a modulating coupling in Eqs. (5) is obtained, e.g., if the frequency ω_1 is replaced by $\omega_1(1 + kx_2)$ and ω_2 by $\omega_2(1 + kx_1)$. The results appear quite analogous to those obtained for the phase oscillators with a modulating coupling in Fig. 3 (the plots are not shown).

Thus, non-normality and non-whiteness do not lead to any change in the methods performance as compared to Sects. 3.1 and 3.2. Probably, this is because the distributions of the phase noises and phase increments are still well-localized in the above examples and do not exhibit any signs of heavy tails that would be a "harder" deviation from normality and lies beyond the scope of this work. Anyway, the examples with the Roessler systems justify an opinion that the conditions for applicability and relative superiority of both methods obtained in Sects. 3.1 and 3.2 are rather general. These conditions are summarized and discussed below.

4 Discussion and conclusions

The suggested PIC method complements the skills of the MPC approach in that the former is sensitive also to such couplings which do not increase a degree of synchrony between the oscillators under study. This is because PIC depends on the presence of common terms in the right-hand sides of the phase dynamics equations and does not strongly depend on the resulting change in the instantaneous phase difference Φ distribution. The PIC method carries additional information about the phase dynamics of the oscillators: positive or negative PICs evidence how the velocities of the oscillators mutually evolve in time. In addition to the detection of coupling at the above indicated optimal τ, the values of PIC for different τ's give a kind of unfolded representation of such mutual evolution versus the time scale. It is also convenient that the PIC method works without any changes for couplings which increase the generalized higher-order $n{:}m$ MPC index, where the MPC method would require checking different trial $n{:}m$ orders from a certain range. Such examples are not illustrated above because the results appear quite similar to the 1:1 case.

The second important circumstance is that the null hypothesis of uncoupled systems in the PIC method is less restrictive in respect of the oscillators properties than that in the MPC method. In the PIC method, one presumes that the phase increments $\Delta\phi_k(t)$ are stationary processes. They must also follow Gaussian distribution (which is not so compulsory as shown in Sect. 3.3) and their autocorrelations must decay over an interval significantly less than the time series length (less than a quarter of the time series length would be enough) which is met in all examples above and typical in practice. The null hypothesis of the MPC method involves additional harder restrictions. Namely, the individual phase dynamics must be described by the equations (1) that implies, in particular, the absence of phase nonlinearities (which is important), accurate estimation of the frequency mismatch and noise levels (also important), and Gaussian white noises (not as important). Due to such restrictive requirements, there are situations where the MPC method is no longer reliable while the PIC method performs well. These are short time series (e.g. 10–30 basic periods in the above examples), strong couplings inducing a high degree of phase synchrony (e.g. Figs. 1a,b), and essential phase nonlinearities (e.g. $b = 0.7$ in Eqs. (4), Figs. 3c,d).

It is worth noting that the requirement of the first-order phase dynamics equations is often met in practice, but sometimes can be strongly violated, e.g. in some radio-engineering systems [27]. If it is violated, the formulas for the MPC significance test do not apply since they are based essentially on the particular form of Eqs. (1). Thus, for higher-order equations all the derivations must be done from the beginning, which is not a straightforward task. The PIC method in such a case is also, strictly speaking, inapplicable because the phase increments (which are the first-order differences) may appear non-stationary for the phase equations of an order $q > 1$. However, the PIC method is easily adjusted via replacing the first differences $\Delta\phi_k(t)$ by the differences of the expected qth order $\Delta^q\phi_k(t)$. An appropriate order q can be determined by a visual inspection of the $\Delta^q\phi_k(t)$ time series: non-stationary processes would exhibit Borwnian-like fluctuations, while stationary ones would be uniformly bounded [24]. No other changes in the entire formalism of the PIC method are necessary. Thereby, the domain of applicability of the PIC method is further extended as compared to the MPC method. Yet, we do not forget that for the widespread case of synchronizing couplings (without the above difficulties) the MPC method is much more sensitive and, probably, its sensitivity is best possible for quite a wide range of systems.

We have focused here on the methods with analytic tests for significance because they are highly desirable in practice. An analytic test allows us to avoid extensive computations and insufficient data problems related to surrogate data-based approaches, especially taking into account that any kind of surrogate data is also constructed under a certain specific null hypothesis: e.g. the shuffling of 2π-intervals within the phase time series [19] relates to the hypothesis of the first-order phase equations.

To conclude, we suggest a method for detection of coupling between two oscillators from a time series with an analytic test for significance. It is based on the computation of the phase increments correlation (PIC) and complements the well-known approach based on the mean phase coherence (MPC) estimate. The suggested PIC method appears more sensitive to non-synchronizing couplings and applicable to a wider range of situations including strong individual phase nonlinearities and higher-order phase dynamics equations. It is also very simple in its implementation. All that suggests that the PIC method may appear a useful complementary tool in studies of the mutual dynamics of oscillatory systems in practice.

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References

1. A.S. Pikovsky, M.G. Rosenblum, J. Kurths, *Synchronization: A universal concept in nonlinear sciences* (Cambridge University Press, Cambridge, 2001)
2. V.S. Anishchenko, V.V. Astakhov, A.B. Neiman, N.E. Vadivasova, L. Schimanski-Geier, *Nonlinear dynamics of chaotic and stochastic systems* (Springer-Verlag, Berlin, 2002)
3. E. Mosekilde, Yu. Maistrenko, D. Postnov, *Chaotic synchronization: Applications to living systems* (World Scientific, Singapore, 2002)
4. S. Boccaletti, J. Kurths, G. Osipov, D. Valladares, C. Zhou, Phys. Rep. **366**, 1 (2002)
5. G.V. Osipov, J. Kurths, C. Zhou, *Synchronization in oscillatory networks* (Springer-Verlag, Berlin, 2007)
6. A. Balanov, N. Janson, D. Postnov, O. Sosnovtseva, *Synchronization: From simple to complex* (Springer-Verlag, Berlin, 2008)
7. F. Mormann, K. Lehnertz, P. David, C.E. Elger, Physica D **144**, 358 (2000)
8. M.G. Rosenblum, A.S. Pikovsky, J. Kurths, C. Schaefer, P.A. Tass, in *Neuro-informatics*, edited by F. Moss and S. Gielen, Handbook of Biological Physics **4** (Elsevier Science, New York, 2000), p. 279
9. A. Kraskov, T. Kreuz, R. Quian Quiroga, P. Grassberger, F. Mormann, K. Lehnertz, C.E. Elger, Epilepsia **43**, 48 (2002)
10. M.D. Prokhorov, V.I. Ponomarenko, V.I. Gridnev, M.B. Bodrov, A.B. Bespyatov, Phys. Rev. E **68**, 041913 (2003)
11. A.Ye. Hramov, A.A. Koronovskii, Chaos **14**, 603 (2004)
12. A.N. Pavlov, O.V. Sosnovtseva, O.N. Pavlova, E. Mosekilde, N.-H. Holstein-Rathlou, Physiol. Meas. **29**, 945 (2008)
13. A.S. Karavaev, M.D. Prokhorov, V.I. Ponomarenko, A.R. Kiselev, V.I. Gridnev, E.I. Ruban, B.P. Bezruchko, Chaos **19**, 033112 (2009)
14. D.A. Smirnov, U.B. Barnikol, T.T. Barnikol, B.P. Bezruchko, C. Hauptmann, C. Buehrle, M. Maarouf, V. Sturm, H.-J. Freund, P.A. Tass, Europhys. Lett. **83**, 20003 (2008)
15. I.I. Mokhov, D.A. Smirnov, Geophys. Res. Lett. **33**, L03708 (2006)
16. B.P. Bezruchko, D.A. Smirnov, *Extracting knowledge from time series: An introduction to nonlinear empirical modeling* (Springer-Verlag, Berlin, 2010)
17. C. Allefeld, J. Kurths, Int. J. Bif. Chaos **14**, 405 (2004)
18. B. Schelter, M. Winterhalder, J. Timmer, M. Peifer, Phys. Lett. A **366**, 382 (2007)
19. J. Brea, D.F. Russell, A.B. Neiman, Chaos **16**, 026111 (2006)
20. D.A. Smirnov, E.V. Sidak, B.P. Bezruchko, Sov. Tech. Phys. Lett. **39**, 40 (2013)
21. P.A. Tass, *Phase resetting in medicine and biology. Stochastic modelling and data analysis* (Springer-Verlag, Berlin, 1999)
22. A.S. Pikovsky, M.G. Rosenblum, J. Kurths, Int. J. Bifurc. Chaos **10**, 2291 (2000)
23. A.A. Borovkov, *Mathematical statistics* (Fizmatlit, Moscow, 2007)
24. G.E.P. Box, G.M. Jenkins, *Time series analysis. Forecasting and control* (Holden-Day, San Francisco, 1970)
25. B. Kralemann, L. Cimponeriu, M. Rosenblum, A. Pikovsky, R. Mrowka, Phys. Rev. E **77**, 066205 (2008)

26. V.S. Anishenko, T.Ye. Vadivasova, G.A. Okrokvertskhov, G.I. Strelkova, Physics – Uspekhi **175**, 163 (2005)
27. V.V. Matrosov, V.D. Shalfeev, *Dynamical chaos in phase systems. A tutorial* (Nizhny Novgorod State University, Nizhny Novgorod, 2007)