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Reconstruction of the Coupling Architecture in an Ensemble of Coupled Time-Delay Systems

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Abstract—A method for reconstructing the coupling architecture and values in an ensemble of time-delay interacting systems with an arbitrary number of couplings between ensemble elements is proposed. This method is based on reconstruction of the model equations of ensemble elements and diagnostics of the coupling significance by successive trial exclusion or adding coupling coefficients to the model.

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The behavior of complex systems of different natures composed of several interacting elements depends on not only the properties of these elements, but also on the character of their coupling. For example, the structure and intensity of couplings in an ensemble of oscillating systems determines the possibility of their synchronization and the formation of different spatial and temporal structures [1–4]. In recent years, much attention has been paid to the problem of revealing the presence of couplings in ensembles of multielement systems and determining their structure and characteristics from time series [5–7]; this problem is solved using methods based on simulation of the phase dynamics. Here, we propose a method for reconstructing the coupling architecture and values in large ensembles of interacting systems based on a method for reconstructing the model equations of ensemble elements. Using this method, we reconstructed for the first time the a priori unknown architecture of couplings in a large ensemble of chaotic time-delay systems with a complex coupling structure.

Let us consider an ensemble composed of diffusively coupled time-delay systems; each of them is described by the equation

$$\varepsilon_i\dot{x}_i(t) = -x_i(t) + f_i(x_i(t-\tau_i)) + \sum_{j=1(j\neq i)}^{M} k_{i,j}(x_j(t)-x_i(t)), \qquad (1)$$

where $i = 1, \ldots, M$; M is the number of elements in the ensemble; ε_i is the inertia parameter; τ_i is the delay time; f_i is a nonlinear function; and $k_{i,j}$ is the coupling coefficient.

To determine all coupling coefficients in the ensemble, we propose a method based on reconstructing model equation (1) for each element in time series. First, we reconstruct the delay times. We established previously that there are hardly any extrema spaced by the delay time in the time realizations of isolated ($k_{j,j} = 0$) time-delay systems of type (1) [8]. Having determined, for different τ values, number N_i of situations where the points of a chaotic time series, spaced by time interval τ, are simultaneously extremal and having plotted dependence $N_i(\tau)$, one can find delay time τ_i as a value at which this dependence exhibits an absolute minimum [8]. The validity of this method for reconstructing τ_i in a chain of coupled time-delay systems was substantiated in [9]. According to our results, this method for determining the delay time remains efficient for ensembles of systems (1) with an arbitrary number of couplings between elements, provided that the interaction of systems does not induce a large number of additional extrema in chaotic time realizations of their oscillations. This assumption remains valid for weak coupling ($k_{i,j} \ll x_i$) even in the case of globally coupled systems (1). Note that the condition of weak coupling and the absence of synchronization between ensemble elements is necessary for almost all methods for revealing couplings [7].

Having determined τ_i, one can reconstruct parameter ε_i, nonlinear function f_i, and the coupling coefficients $k_{i,j}$ of the ith time-delay system (1), knowing the time series of oscillations of all ensemble elements. To this end, we propose the following approach. Let us write Eq. (1) in the form

$$\varepsilon_i\dot{x}_i(t) + x_i(t) - \sum_{j=1(j\neq i)}^{M} k_{i,j}(x_j(t)-x_i(t)) = f_i(x_i(t-\tau_i)). \qquad (2)$$

If one plots the dependence of the left-hand side of Eq. (2) on $x_i(t - \tau_i)$, it will coincide with the function f_i. Since ε_i and $k_{i,j}$ are unknown beforehand, they will be sought by minimizing the parameter

$$L_i(\varepsilon_i, k_{i,j}) = \sum_{n=1}^{S-1}((y_{i,n+1} - y_{i,n})^2 + (z_{i,n+1} - z_{i,n})^2). \quad (3)$$

This parameter characterizes the distance between the points in the (y_i, z_i) plane, which are sorted with respect to the coordinate y_i. Here,

$$y_i = x_i(t - \tau_i),$$

$$z_i = \varepsilon_i \dot{x}_i(t) + x_i(t) - \sum_{j=1(j \neq i)}^{M} k_{i,j}(x_j(t) - x_i(t)),$$

n is the point number and S is the number of points. When the ε_i and $k_{i,j}$ values are chosen incorrectly, the points in the (y_i, z_i) plane cannot be connected by one-dimensional curve f_i; therefore, the $L_i(\varepsilon_i, k_{i,j})$ value is larger than that for true ε_i and $k_{i,j}$.

For ε_i and $k_{i,j}$, we set starting conjectures and then refine them by the simplex method [10] with minimization of (3) (its minimum is referred to as $L_{i,M}$). At $M \leq 4$ and in the absence of noise, all parameters are reconstructed with a high accuracy. However, at $M > 4$, the situation in which the method does not allow one to reveal the absent couplings between ensemble elements becomes typical. These couplings are identified as weak because of the presence of indirect couplings via other elements. Insignificant couplings can be rejected by the method of successive trial exclusion of the coefficients $k_{i,j}$ from model (1). We hypothesize that the two elements are not coupled (by excluding the corresponding coupling coefficient $k_{i,j}$) and reconstruct the other parameters of the model, finding the minimum $L_{i,j,M-1}$ of function (3). This procedure is then repeated by excluding another $k_{i,j}$ at a fixed i, etc., for all $j = 1, \ldots, M$ $(j \neq i)$. Finally, we determine the exclusion of which $k_{i,j}$ yields $L_{i,M-1} = \min\limits_j L_{i,j,M-1}$ and estimate the statistical significance of the magnitude $L = L_{i,M-1}/L_{i,M}$ based on the following considerations. At large S, the differences $y_{i,n+1} - y_{i,n}$ and $z_{i,n+1}$ in (3) are distributed according to the close-to-normal law; here, $S/2$ of points can be considered as independent because they have no common coordinates. In addition, $L_{i,M}$ depends on M parameters of model (2), a fact that reduces the number of independent values in (3) to $S/2 - M$. Then, taking into account that the sum of K squared independent normally distributed values obeys the chi-square law with K degrees of freedom [11], we find that the $L_{i,M}$ values obtained at different parameters and/or in the presence of noise are distributed according to the chi-square law with $S/2 - M$ degrees of freedom and the $L_{i,M-1}$ values are distributed according to the chi-square law with $S/2 - M + 1$ degrees of freedom.

The parameter X, which is a ratio of two independent random values distributed according to the chi-square law with v and w degrees of freedom, is known to obey the Fisher–Snedecor distribution with the distribution function

$$F_{v,w}(X) = B_d\left(\frac{v}{2}, \frac{w}{2}\right), \quad (4)$$

where B is the incomplete β function and $d = vX/(vX + w)$ [12]. Therefore, the parameter L is described by distribution function (4) with $X = L$, $v = S/2 - M + 1$, and $w = S/2 - M$. We denote the L value at which $F_{v,w}(L_{1-p}) = 1 - p$ as L_{1-p} (p is the statistical-significance level). Then, if $L > L_{1-p}$, one can conclude at the significance level p that the elements are coupled and, accordingly, all $k_{i,j} \neq 0$. In the opposite case, we conclude that the corresponding elements are not coupled and check the significance level for other couplings, successively excluding one of the remaining couplings of the ith element. The procedure is repeated until all couplings become significant. This approach makes it possible to reconstruct the coupling architecture, parameters of all elements, and their nonlinear functions.

If the number of couplings between ensemble elements is known to be small, the method of successive trial addition of the coefficients $k_{i,j}$ to model (1) is preferred for reconstructing the coupling architecture and values. First, we find the minimum $L_{i,1}$ of function (3) on the assumption that all $k_{i,j}$ are absent in Eq. (1) (i.e., there are no couplings). We then successively (one by one) introduce $k_{i,j}$ into (1) to find the minimum $L_{i,j,2}$ of function (3). Having enumerated all $j \neq i$, we find $L_{i,2} = \min\limits_j L_{i,j,2}$. If $L > L_{1-p}$ $(L = L_{i,1}/L_{i,2})$ and $F_{v,w}$ is plotted at $v = S/2 - 1$ and $w = S/2 - 2$, the coupling introduced is nonzero at significance level p. This procedure is repeated until another coupling added to the model turns out to be insignificant.

As an example, we will consider the reconstruction of the coupling architecture in an ensemble of coupled Mackey–Glass systems described by Eq. (1) with the function

$$f_i(x_i(t - \tau_i)) = \frac{a_i x_i(t - \tau_i)}{b_i(1 + x_i^{10}(t - \tau_i))} \quad (5)$$

and $\varepsilon_i = 1/b_i$. These systems are affected by independent normal noise $\xi_i(t)$ with a zero mean and dispersion σ_t^2. The parameters of the elements were specified randomly: integer $\tau_i \in [300, 400]$, $\varepsilon_i \in [8, 12]$, $a_i \in [0.2, 0.25]$, $k_{i,j} \in [0.08, 0.12]$, and $\sigma_i^2 = 10^{-4}$. All elements oscillate chaotically. The time series of each element contains 10 000 points with a sample step of 1.

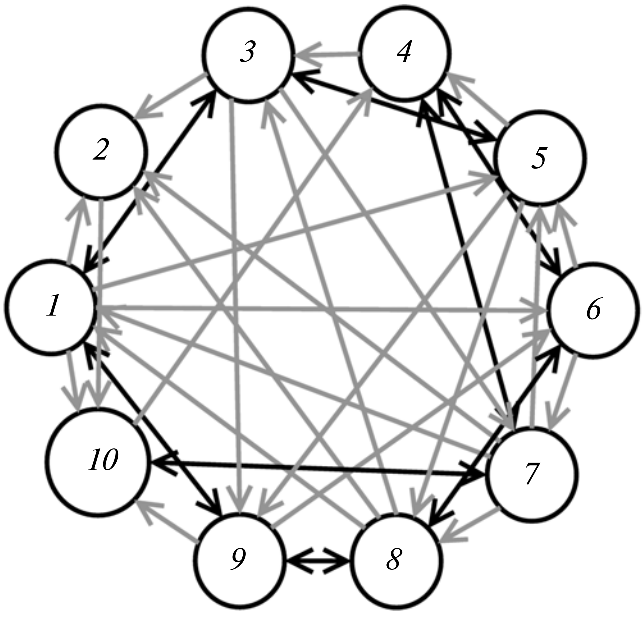

Fig. 1. Coupling architecture in an ensemble of ten elements. Forty of ninety possible couplings are present. Bidirectional and unidirectional couplings are shown in black and gray, respectively.

Figure 1 shows the architecture of randomly chosen couplings in an ensemble of $M = 10$ elements.

The results of reconstruction of one element with the parameters $\tau_5 = 348$, $\varepsilon_5 = 9.5$, $k_{5,1} = 0.112$, $k_{5,3} = 0.085$, $k_{5,6} = 0.116$, $k_{5,7} = 0.090$, and $k_{5,j} = 0$ ($j = 2$, 4, 5, 8, 9, and 10) are shown in Fig. 2. Having calculated number N_5 of times when $\dot{x}_5(t)$ and $\dot{x}_5(t-\tau)$ simultaneously turn to zero for different τ, enumerated with a step of 1, we plot the dependence $N_5(\tau)$ (Fig. 2a). To estimate the time derivative from time series, we used a local parabolic approximation. The minimum $N_5(\tau)$ is observed at the true delay time $\tau = \tau_5 = 348$.

Figure 2b shows the function f_5 (in gray) obtained after reconstructing Eq. (1) on the assumption that all ensemble elements are uncoupled and the function f_5 (in black) reconstructed using the method of successive trial addition of the coupling coefficients to the model at $p = 0.05$. Taking into account the coupling architecture significantly improves the quality of reconstruction of the nonlinear function and the accuracy of estimating the model parameters. The errors in reconstructing the coupling architecture are mainly caused by the presence of noise.

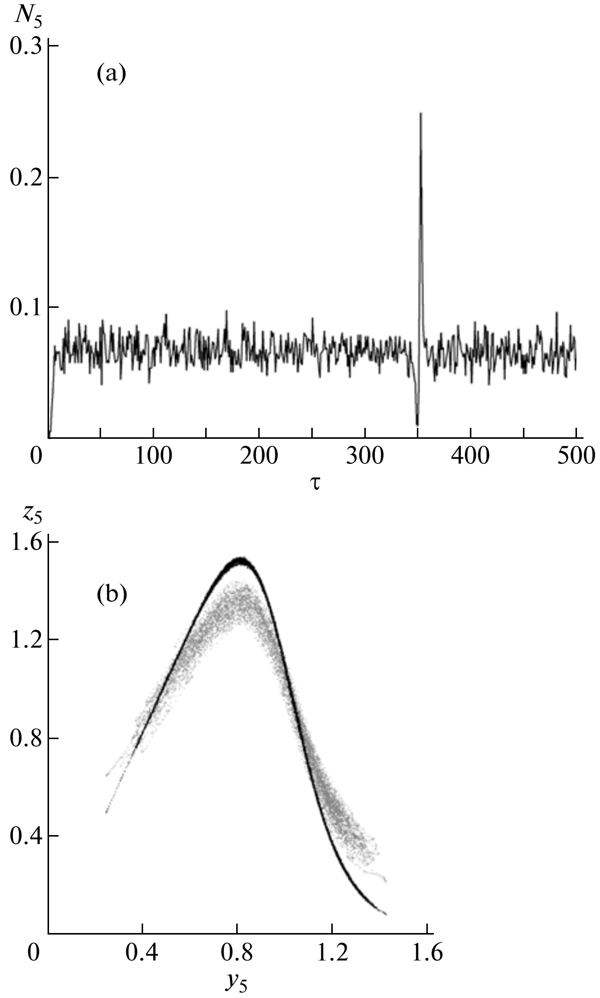

Fig. 2. (a) Number N_5 of pairs of extrema in the time series of the variable $x_5(t)$ (spaced by τ) normalized to the total number of extrema in the series: $N_{5\min}(\tau) = N_5(348)$. (b) Nonlinear function f_5 reconstructed in the (y_5, z_5) plane with $y_5 = x_5(t - \tau'_5)$ and $z_5 = \varepsilon'_5\dot{x}_5(t) + x_5(t) - \sum_{j=1(j\neq 5)}^{10} k'_{5,j}(x_j(t) - x_5(t))$ at the following parameters obtained during reconstruction: (gray) $\tau'_5 = 348$, $\varepsilon'_5 = 8.4$, and $k'_{5,j} = 0$ ($j = 1, \ldots, 10; j \neq 5$) and (black) $\tau'_5 = 348$, $\varepsilon'_5 = 9.6$, $k'_{5,1} = 0.111$, $k'_{5,3} = 0.085$, $k'_{5,6} = 0.105$, $k'_{5,7} = 0.080$, and $k'_{5,j} = 0$ ($j = 2, 4, 5, 8, 9, 10$).

The results of reconstructing the coupling architecture in the entire ensemble, obtained using the method of adding couplings, are shown in Fig. 3. A square with the coordinates (j,i) illustrates the influence of the jth element on the ith element (except for the squares in the diagonal, which carry no information). At the significance level $p = 0.05$, we found 39 of the 40 existing couplings (Fig. 1). Only one coupling was missed, and spurious couplings were absent. Note that, when reconstructing the coupling architecture based on the same time series using the method of excluding couplings, we obtained many spurious couplings at the same p. The method of excluding couplings is more efficient for reconstructing ensembles where the number of existing couplings is many times larger than the number of absent couplings. On the contrary, the method of adding couplings is most efficient when the number of couplings in an ensemble is small, even in

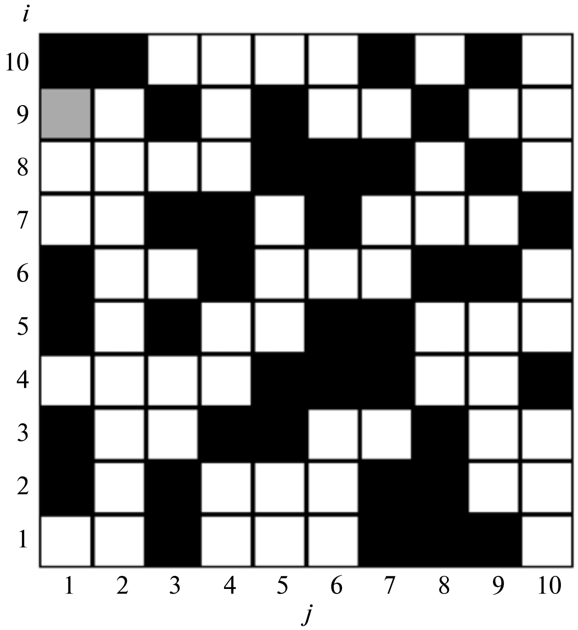

Fig. 3. Diagram of the results of reconstructing the coupling architecture in an ensemble of ten Mackey–Glass systems obtained at the significance level $p = 0.05$ using the method of successive trial addition of coupling coefficients to the model. Correctly reconstructed couplings, correctly found absent couplings, and missed couplings are shown in black, white, and gray, respectively.

the case of large ensembles ($M = 50$). The case under consideration, in which the numbers of existing and absent couplings are comparable, is the most complex for reconstruction. In these situations, the method of adding couplings is more efficient.

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