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Estimating Characteristics of Ensemble of Coupled Delay-Feedback Systems from Their Experimental Time Series

V. I. Ponomarenko and M. D. Prokhorov*

Institute of Radio Engineering and Electronics (Saratov Branch), Russian Academy of Sciences, Saratov, 410019 Russia
**e-mail: mdprokhorov@yandex.ru*

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Abstract—Model delay-differential equations for an ensemble of coupled delay-feedback systems have been reconstructed for the first time from their experimental time series.

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In recent years, model ensembles of coupled delay-feedback systems have been frequently used for descriptions and simulations of processes in various physical [1–3], chemical [4, 5], and biological systems [6–8]. An important task in this approach consists in estimating parameters of the ensemble of interacting systems from their experimental time series. Solving this task makes it possible to predict the behavior of systems in the case of modification of these parameters, allows the adequacy of notions about the modeled object to be assessed, provides a basis for the classification of coupled systems and their functioning regimes, and determine parameters that are inaccessible for direct measurement in experiments.

Most of the previously known methods of reconstruction of the model equations of time-delay systems from their time series were aimed at the recovery of isolated systems [9–17]. The presence of coupling between subsystems in an ensemble introduces some specific features in the problem of their recovery and requires developing new approaches [18]. To the best of the authors' knowledge, recovery of the equations of dynamics from time series for an ensemble of real time-delay systems has not been reported until now.

This Letter reports on the first solution of this problem. Methods proposed in our previous investigations are generalized to the case of ensembles with an arbitrary number of coupled time-delay systems and are applied for the first time to reconstructing the parameters of an ensemble of coupled oscillators with delayed feedback from their experimental time series.

Consider an ensemble consisting of three coupled radio-frequency oscillators with delayed feedback. Figure 1 shows a block scheme of an experimental setup involving three coupled ring oscillators, each comprising a delay line, a nonlinear element, and a low-frequency first-order *RC* filter. The nonlinear elements and delay lines are implemented on digital components, while the filters are analog devices. The digital and analog elements of this scheme are linked via the corresponding analog-to digital (ADC) and digital-to-analog (DAC) converters. The oscillators are coupled via resistors R_c.

A model equation for the ith subsystem in the chain is as follows:

$$R_i C_i \dot{V}_i(t) = -V_i(t) + f_i(V_i(t-\tau_i)) + \frac{R_i}{R_c}[V_{i+1}(t) - 2V_i(t) + V_{i-1}(t)], \quad (1)$$

where $V_i(t)$ and $V_i(t-\tau_i)$ is the input and output voltage, respectively, of the ith delay line; τ_i is the delay time; R_i and C_i is the resistance and capacitance, respectively, of the ith filter; and f_i is the transmission function of the ith nonlinear element. The periodic boundary conditions are set as $x_4 = x_1$. It is convenient to introduce the quantities $\varepsilon_i = R_iC_i$ and $k_i = R_i/R_c$, so

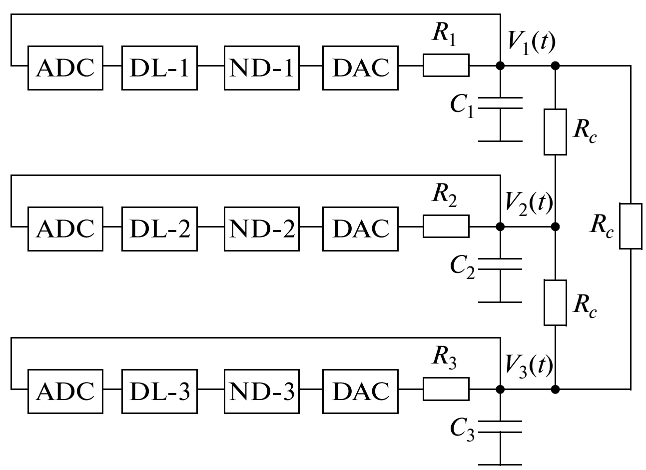

Fig. 1. Schematic diagram of the experimental setup: (DL-1, DL-2, DL-3) delay lines; (ND-1, ND-2, ND-3) nonlinear elements; (ADC) analog-to digital converters; (DAC) digital-to-analog converters.

that ε_i characterizes the inertial properties and k_i determines the coupling level.

Experiments were performed in a system with the following values of parameters: $\tau_1 = 13.6$ ms; $\tau_2 = 16.4$ ms; $\tau_3 = 20.4$ ms; $\varepsilon_1 = 2.88$ ms; $\varepsilon_2 = 2.91$ ms; $\varepsilon_3 = 2.94$ ms; and $k_{1, 2, 3} = 0.1$. Chaotic $V_i(t)$ signals were measured and recorded using a three-channel DAC with a sampling frequency of $f_s = 10$ kHz. All nonlinear elements had quadratic transmission function f_i. Figure 2a shows a typical time series of $V_2(t)$.

Previously, it was established that time series of the isolated ($k_i = 0$) delay-feedback systems of type (1) contain virtually no extrema separated from each other by an interval equal to the delay time [13]. Then, it is necessary to find extrema in a recorded time series, determine the number N of the pairs of extrema spaced from each other by various intervals τ in this time series, construct the $N(\tau)$ plot, and estimate the delay time τ_i as the position of the absolute minimum of this function [13]. Now we will show that this method of recovering the delay time can also be applied to coupled time-delay systems described by Eq. (1). Differentiating Eq. (1) with respect to t yields

$$\varepsilon_i \ddot{V}_i = -\dot{V}_i(t) + \frac{df_i(V_i(t-\tau_i))}{dV_i(t-\tau_i)}\dot{V}_i(t-\tau_i) + k_i[\dot{V}_{i+1}(t) - 2\dot{V}_i(t) + \dot{V}_{i-1}(t)]. \quad (2)$$

For $\dot{V}_i(t) = 0$, in the typical case of quadratic extrema we have $\ddot{V}_i(t) \neq 0$. Then, for $\varepsilon_i \neq 0$, we obtain the condition

$$\frac{df_i(V_i(t-\tau_i))}{dV_i(t-\tau_i)}\dot{V}_i(t-\tau_i) + k_i[\dot{V}_{i+1}(t) + \dot{V}_{i-1}(t)] \neq 0 \quad (3)$$

which is valid provided that $\dot{V}_i(t - \tau_i) \neq 0$ and/or $k_i[\dot{V}_{i+1}(t) + \dot{V}_{i-1}(t)] \neq 0$. The latter inequality never holds in the absence of coupling (since $k_i = 0$) and in the case of strong coupling that ensures the synchronization of subsystems, since $\dot{V}_{i+1}(t) = \dot{V}_{i-1}(t) = \dot{V}_i(t)$ and $\dot{V}_i(t) = 0$ in deriving condition (3). Therefore, the first term in (3) in these boundary cases is nonzero and, hence the derivatives $\dot{V}_i(t)$ and $\dot{V}_i(t - \tau_i)$ do not vanish simultaneously. This implies that no other extrema can exist in the time series of $V_i(t)$ at a distance of τ_i from any quadratic extremum. In the intermediate case of weak and moderate coupling, there is some probability to find pairs of extrema spaced by τ_i. However, investigations show that, in the general case, this probability is much smaller than that of finding a pair of extrema spaced by $\tau \neq \tau_i$. As a result, the $N_i(\tau)$ plot will exhibit a minimum at $\tau = \tau_i$ in a broad range of coupling coefficients.

Thus, we count the numbers N_2 of cases where $\dot{V}_2(t)$ and $\dot{V}_2(t - \tau)$ in the time series (Fig. 2a) simultaneously vanish for various values of τ run at a step of $T_s = 0.1$ ms and plot $N_2(\tau)$ normalized to the total number of extrema in the time series (Fig. 2b). In order to evaluate the derivative from the time series, we used a parabolic approximation. The absolute minimum in $N_2(\tau)$ is observed at $\tau = 16.4$ ms (Fig. 2b).

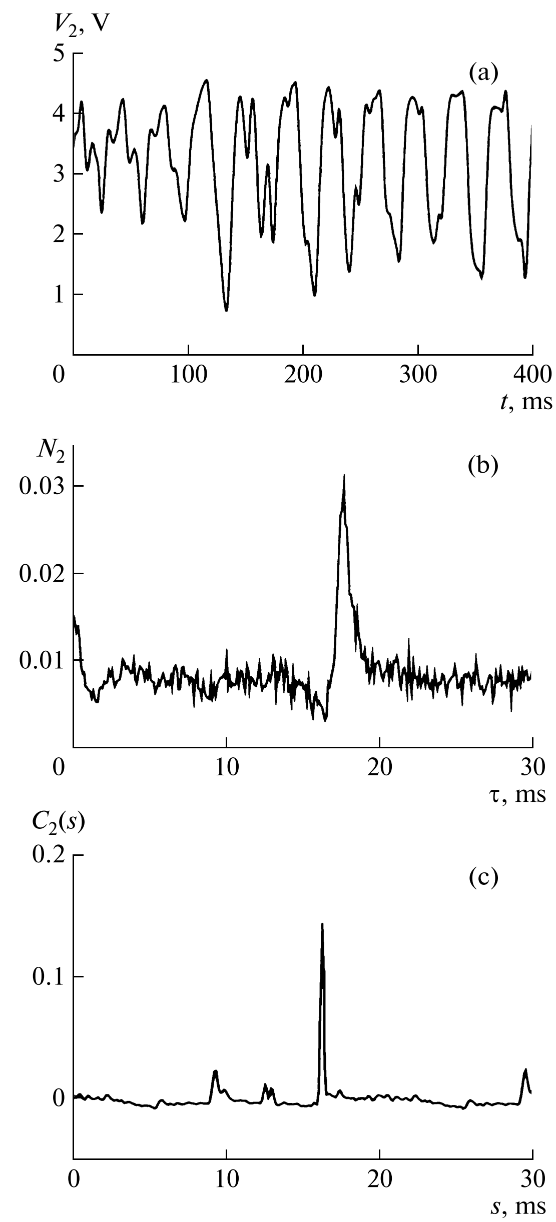

Fig. 2. Reconstruction of the delay time: (a) experimental chaotic time series $V_2(t)$ of the second oscillator; (b) plot of the number N_2 of the pairs of extrema in the given time series spaced by various time intervals τ (normalized to the total number of extrema), which shows that $N_{2\min}(\tau) = N_2(16.4\text{ ms})$; (c) mutual correlation function (4) for an ensemble with subsystems performing periodic oscillations in the absence of external excitation, which shows that $C_{2\max}(s) = C_2(16.3\text{ ms})$.

If a time-delay system performs periodic oscillations, the above method cannot be used to determine τ_i because extrema in the time series are situated peri-

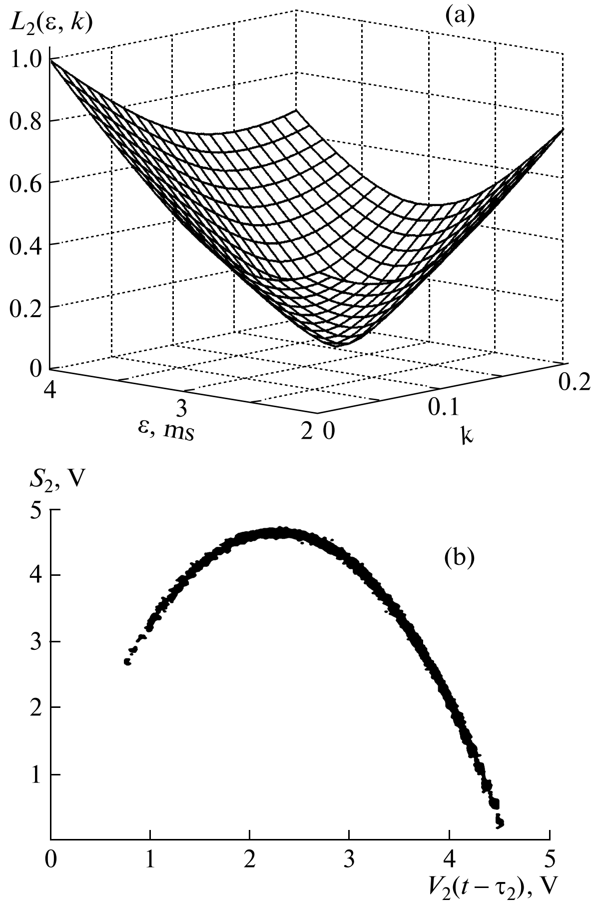

Fig. 3. Reconstruction of parameters of the second subsystem: (a) plot of $L_2(\varepsilon, k)$ (normalized to the maximum value in a given interval of parameters), which yields $L_{2\min}(\varepsilon, k) = L_2(2.75\ \text{ms}, 0.09)$; (b) nonlinear transmission function $S_2 = \varepsilon_2' \dot{V}_2(t) + V_2(t) - k_2'[V_3(t) - 2V_2(t) + V_1(t)]$ reconstructed for $\tau_2 = 16.4$ ms, $\varepsilon_2' = 2.75$ ms, and $k_2' = 0.09$.

odically. For isolated time-delay systems occurring in a regime or periodic oscillations, we have recently proposed a method of delay time recovery, which is based on an analysis of the system response to a weak external perturbation [16]. According to this, if the $V_i(t)$ variable of an isolated time-delay system is driven by an external signal $y_i(t)$ of rectangular shape, then the mutual correlation function defined as

$$C_i(s) = \frac{\langle |\ddot{y}_i(t)| |\ddot{V}_i(t+s)| \rangle}{\sqrt{\langle |\ddot{y}_i(t)|^2 \rangle \langle \ddot{V}_i(t) \rangle^2}}, \quad (4)$$

(where angle brackets denote averaging with respect to time) will have a clearly pronounced maximum at $s = \tau_i$. This method allows rectangular pulses of small amplitude to be used.

Let us study the possibility of using this method for determining the delay time in an ensemble of coupled oscillators. Consider the excitation of the ith subsystem by an external signal $y_i(t)$ such that the model equation takes the following form:

$$\begin{aligned} \varepsilon_i \dot{V}_i(t) &= -V_i(t) + f_i(V_i(t-\tau_i) + y_i(t-\tau_i)) \\ &\quad + k_i[V_{i+1}(t) - 2V_i(t) + V_{i-1}(t)]. \end{aligned} \quad (5)$$

Perturbation $y_i(t)$ has the form of rectangular pulses with amplitude A_i, period T_i, and duration M_i. Let the second oscillator be excited by an external signal $y_2(t)$ introduced via a summing amplifier between filter and delay line. The parameters of oscillators are selected such that they perform periodic oscillations in the absence of external perturbations.

Figure 2c shows the mutual correlation function (4) calculated for a perturbing pulsed signal; with amplitude $A_2 = 0.05$ V, period $T_2 = 40$ ms, and duration $M_2 = T_2/2$. For a step of $s = 0.1$ ms, $C_2(s)$ exhibits the main peak at $s = 16.3$ ms, which demonstrates that the delay time is recovered with a high precision. The proposed method can be applied not only to systems occurring in the regime of periodic oscillations, but also to systems performing chaotic oscillations. Moreover, it can be used for estimating the delay time in noisy systems, and the admissible noise levels is higher that for the method based on the analysis of extrema in the time series.

Once the delay time of the ith subsystem in an ensemble of coupled oscillators with delayed feedback is determined, we can also recover the corresponding values of parameters ε_i and k_i and the nonlinear function f_i from time series of the ith and other coupled oscillators. This can be achieved using the following approach. Rewriting Eq. (1) as

$$\begin{aligned} \varepsilon_i \dot{V}_i(t) + V_i(t) - k_i[V_{i+1}(t) - 2V_i(t) + V_{i-1}(t)] \\ = f_i(V_i(t-\tau_i)) \end{aligned} \quad (6)$$

one can infer that a plot of $\varepsilon_i \dot{V}_i(t) + V_i(t) - k_i[V_{i+1}(t) - 2V_i(t) + V_{i-1}(t)]$ versus $V_i(t-\tau_i)$ reproduces the nonlinear function f_i. Since the parameters ε_i and k_i are unknown, we suggest trying various values from certain intervals so as to obtain single-valued relationships on the corresponding planes, which is only possible provided a correct choice of these parameters. As a quantitative criterion of such a unique relationship in the search for ε_i and k_i, we can use the minimum length $L_i(\varepsilon, k)$ of a broken line connecting sequential points (ordered with respect to the abscissa) on the corresponding planes. Then, a minimum in $L_i(\varepsilon, k)$ is observed provided a correct choice of the parameters, and the corresponding set of points on the plane will reproduce the nonlinear function, which, if necessary, can be appropriately approximated.

Let us apply the proposed approach to the experimental chaotic time series for an ensemble with the parameters specified above. The plot of $L_2(\varepsilon, k)$ constructed for the recovered delay time of $\tau_2 = 16.4$ ms (see Fig. 2b) and the values of ε and k tried at an 0.01

step, exhibits a minimum at $\varepsilon = 2.75$ ms and $k = 0.09$ (Fig. 3a). Thus, the proposed procedure provides estimates that are close to ε_2 and k_2. Figure 3b shows a nonlinear function reconstructed from the experimental chaotic time series for the given ensemble. It also well coincides with the true transmission characteristic f_2 of the nonlinear element of the second oscillator. In an analogous manner, we can recover the characteristics of other subsystems of the ensemble.

The proposed method has no limitations with respect to the number of subsystems in the ensemble. Moreover, it can be expanded so as to apply to coupled high-order time-delay systems and systems with several delay times. In the case of synchronization in the ensemble of coupled time-delay oscillators, the proposed method can be used for recovering the parameters of local elements, but the coupling coefficients in this case cannot be determined.

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