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Estimating Delay from a Dynamical System Response to a Weak Pulsed Action

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Abstract—A new method is proposed for estimating the delay time of a delayed-feedback system, which is based on an analysis of the system response to a weak external perturbation in the form of rectangular pulses. The method is applicable to systems that perform both periodic and chaotic oscillations. The efficiency of the proposed procedure is demonstrated based on numerical examples and for the experimental time series of a real radiophysical system.

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Autooscillatory systems with delayed feedback are frequently encountered in nature and widely employed in technology. The interest in these dynamical systems is related to the fact that delayed response and feedback are inherent in many objects and processes in physics, chemistry, biology, and climate [1–4]. In studying delayed-feedback systems, it is important to know the values of delay times, which, to a considerable extent, determine the dynamics of these systems. Various methods have been proposed for recovering the delay times of dynamical systems from the chaotic time series of their oscillations [5–12]. Unfortunately, these methods are not effective in cases where these systems occur in periodic regimes [13]. However, many of the practically important dynamical delayed-feedback systems operate in periodic or nearly periodic regimes [14, 15]. Therefore, the task of developing methods using time series for the recovery of parameters of time-delay systems performing both chaotic and periodic oscillations is of considerable importance.

The first attempts in this direction were made quite recently [16–18] using an approach based on the external perturbation of a delayed-feedback system and an analysis of the response. The proposed methods employed noise [16], regular signals [17], and special perturbations that suppress self-oscillations [18]. In all cases, it was necessary to use external signals of large amplitudes that are comparable with or even exceed that of intrinsic self-oscillations. However, the use of strong external drive signals is not always possible and weak perturbations are preferred. Recently, we have developed an approach [19] that uses the accumulation technique, but the employed signal had a rather involved shape that complicated the practical application of this method.

This Letter describes a new method for estimating the delay time of a delayed-feedback system, which is based on analysis of the system response to a weak pulsed external perturbation of simple shape. The method is applicable to systems that perform both periodic and chaotic oscillations.

Consider a ring autooscillatory system (Fig. 1) with delayed feedback that comprises a delay line, a nonlinear element, and a filter, which performs self-sustained oscillations described by the dynamical variable $x(t)$. Let the system be perturbed by an external signal $y(t)$, which has the form of rectangular pulses with amplitude A, period T, and duration M. The form of a model equation describing this system is determined by the filter parameters and depends on the point at which the external signal is introduced into the ring system with delay. In the case of a low-frequency filter of the first order and the $y(t)$ signal introduced between the filter and delay line (Fig. 1), the autooscillatory system is described by the following

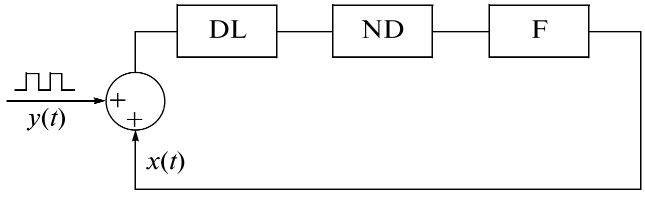

Fig. 1. Schematic diagram of the ring autooscillatory system with delayed feedback, which is driven by an external pulsed signal $y(t)$: (DL) delay line; (ND) nonlinear element; (F) filter.

first-order differential equation with the delayed argument

$$\varepsilon\dot{x}(t) = -x(t) + f(x(t-\tau) + y(t-\tau)), \qquad (1)$$

where τ is the delay time, ε is a parameter characterizing the inertia of the system, and f is the nonlinear function.

As can be seen from Eq. (1), the action of signal $y(t)$ influences the system behavior with delay τ after application. By the same token, termination of the external signal will only be manifested with the same delay. If signal $y(t)$ has the form of rectangular pulses with leading fronts at $t = nT$ and trailing fronts at $T = nT + M$ ($n = 1, 2, ...$), the trajectory $x(t)$ will be perturbed at $t = nT + \tau$ and $t = nT + M + \tau$. These perturbations are manifested by breaks in the time series $x(t)$, which are insignificant provided that the external signal amplitude is small. Changes in the system dynamics are more pronounced in the derivative of $x(t)$ with respect to time. Indeed, the $\dot{x}(t)$ curve exhibits jumps delayed by τ with respect to the leading and trailing fronts of each pulse. The system response to external action is most conveniently analyzed in the form of its second derivative $\ddot{x}(t)$. The time series of this derivative exhibits narrow peaks (dips) delayed by τ with respect to the leading and trailing fronts of each pulse. These peaks are well distinguished, even at small amplitudes of external pulses.

In the analysis of correlations between the external action and system response, one should use signals subjected to identical transformations. In particular, using the cross-correlation function for $\ddot{x}(t)$ and $\ddot{y}(t)$, it is possible to estimate τ. However, since the numerical estimates of $y(t)$ can take both positive and negative values near the pulse front, the delay time will correspond to a zero of the correlation function, which is situated between the main maximum and minimum.

The most convenient form of the cross-correlation function for estimating τ is as follows:

$$C(s) = \frac{\langle|\ddot{y}(t)||\ddot{x}(t+s)|\rangle}{\sqrt{\langle|\ddot{y}(t)|^2\langle|\ddot{x}(t)|\rangle^2}}, \qquad (2)$$

where angle brackets denote averaging with respect to time. Note that $|\ddot{y}(t)|$ takes only positive values at the pulse fronts and $C(s)$ has a clearly pronounced maximum at $s = \tau$ (Fig. 2a).

The proposed method is illustrated below in application to system (1) with $\tau = 800$, $\varepsilon = 20$, and $f(x) = \lambda - x^2$, where λ is the parameter of nonlinearity. In the absence of external perturbations, the system with $\lambda = 1$ performs periodic autooscillations with amplitude $A_a = 1$ and period $T_a = 1638$. Let the external perturbation be a meander with $A = 0.01$, $T = 1900$, and $M = T/2$. Function (2) constructed with a step of $s = 1$ exhibits a large peak at $s = \tau = 800$ (Fig. 2a) and allows the delay time to exactly determined for the meander amplitudes $A \geq 0.002$. The derivatives $\ddot{x}(t)$ and $\ddot{y}(t)$ were evaluated from the time series of $x(t)$ and $y(t)$ using a simple difference scheme. The plot was constructed for 20000 points, but the proposed method allows shorter series to be used as well. For the indicated values of parameters, τ can be accurately estimated using only 3500 points (i.e., two pulses). In the case of a chaotic behavior of system (1), the form of function (2) is also qualitatively similar to that presented in Fig. 2a.

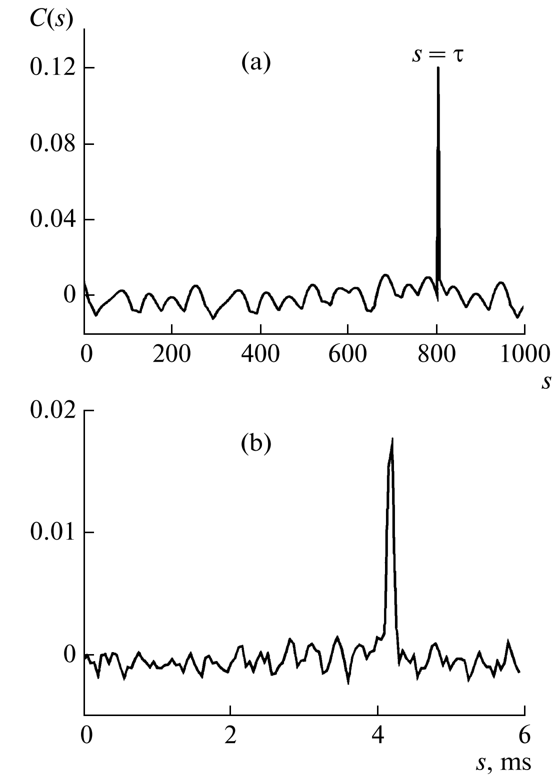

Fig. 2. Cross-correlation function (2) constructed using (a) numerical and (b) experimentally measured data for a system described by Eq. (1), which performs periodic oscillations in the absence of external perturbations.

It should be noted that the number of peaks in $C(s)$ for $s \in [0, \tau]$ is determined by the ratios T/τ and M/T. At $M = T/2$, the delay time corresponds to the first peak in $C(s)$ for $T \geq 2\tau$ (Fig. 2a) and the kth peak for $2\tau/k \leq T \leq 2\tau/(k-1)$ ($k = 2, 3, ...$). As is known, the main mode of periodic oscillations obeys the relation $\tau < T_a/2$, where T_a is the period of autooscillations in the unperturbed delay-feedback system. For this reason, it is recommended to use the external signal with $T \geq T_a$ and $M = T/2$ and estimate τ using the first peak of $C(s)$. If the use of pulses with a shorter period is preferred or it is a priori unknown that the observed signal represents the main mode and, for a system performing chaotic oscillations, one should first perturb the system by a pulsed signal with $T = T_1$, then, using a sig-

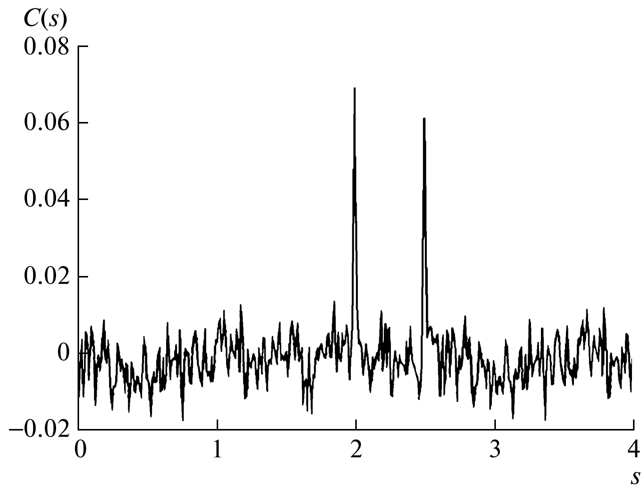

Fig. 3. Cross-correlation function (2) for a chaotic system described by Eq. (3).

nal with $T = T_2$ that is close to T_1, compare the functions $C(s)$ obtained in the two cases. For $M = T/2$, the distance between $C(s)$ peaks is always equal to $T/2$. Hence, the peaks for different T will be observed at different s and only the position of a peak corresponding to the delay time will remain unchanged. Once such a peak is found, the τ value can be estimated.

For $M \neq T/2$, the plot of C(s) will exhibit an additional peak between $s = 0$ and $s = \tau$. However, the peak corresponding to $s = \tau$ will still be maximum and, hence, the M value can be varied within very broad limits. The proposed method allows even very short pulses ($M < T/100$) to be used without increasing their amplitude, which can be useful where the perturbing action must be minimized.

Now let us consider an example of application of the proposed method to experimental time series of a real radiophysical system representing a ring oscillator circuit with delayed feedback, which is driven by an external pulsed signal (Fig. 1). The nonlinear element was an FET-based amplifier with a quadratic transmission characteristic f and the inertia determined by a first-order RC filter with resistance R and capacitance C ($\varepsilon = RC$). The system dynamics is described by Eq. (1), where $x(t)$ and $x(t - \tau)$ are the voltages at the delay line input and output, respectively. In the absence of external perturbations, the oscillator with $\tau = 4.16$ ms and $\varepsilon = 0.46$ ms exhibited periodic autooscillations with an amplitude of $A_a = 1.5$ V and a period of $T_a = 9.2$ ms. Using an analog-to-digital converter with a sampling frequency of $f_s = 20$ kHz, we recorded $x(t)$ and $y(t)$ for $A = 20$ mV, $T = 11.1$ ms, and $M = T/2$. Figure 2b shows the corresponding cross-correlation function (2), which exhibits a maximum at $s = 4.2$ ms. Thus, the delay time is estimated with a high accuracy.

The proposed method can be expanded so as to apply to systems with several delay times. This will be illustrated for a system described by the generalized Ikeda equation, which is obtained by introducing the second delay time in the presence of a dynamic noise as follows:

$$\dot{x}(t) = -x(t) + \lambda[\sin(x(t-\tau_1) + y(t-\tau_1)) + \sin(x(t-\tau_2) + y(t-\tau_2))] + \xi(t). \quad (3)$$

For $\lambda = 10$, $\tau_1 = 2$, and $\tau_2 = 2.5$, this system exhibits a chaotic dynamics. Figure 3 shows a plot of the correlation function (2) for a system under the action of a pulsed signal $y(t)$ with $A = 0.5$, $T = 5.2$, and $M = T/2$ in the presence of a 20% Gaussian white noise $\xi(t)$ with zero mean. At a step of $s = 0.01$, the first two peaks of $C(s)$ are situated at $s = 2.00$ and $s = 2.50$, which shows that both delay times are exactly determined despite a rather high noise level. Moreover, for the indicated values of parameters, the values of τ_1 and τ_2 are exactly determined even at a noise level reaching 40%. Analogous results are obtained by applying this method to system (3) occurring in periodic regimes. Thus, our method is characterized by a high resistance to noise, unlike, e.g., the method proposed in [18], which is only applicable at a noise level below 1%.

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