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VIOLATION OF THE FINAL STATES PROBABILITY SYMMETRY FOR EXPERIMENTAL TIME-DELAY SYSTEM AND COUPLED MAPS WITH TIME-VARYING PARAMETERS

B. P. BEZRUCHKO, V. I. PONOMARENKO, M. D. PROKHOROV
and YE. P. SELEZNEV*
Institute of Radio Engineering and Electronics
of Russian Academy of Sciences, Saratov Branch,
Zelyonaya Str., 38, Saratov 410019, Russia
**sbire@sgu.ru*

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The phenomenon of violation of equal probability of the postbifurcation states under a fast change of the control parameter is studied with an experimental system with continuous time and with a system of coupled maps. It is shown that the choice of one of the possible final states differing only in the phase of oscillations depends on the rate of the control parameter variation and noise level. For the coupled system, the dependence of the final states basin structure on the coupling coefficient is investigated.

Keywords: Multistability; fast bifurcation transitions; basins of attractors.

1. Introduction

Systems with time-varying parameters are widespread in nature. Their investigation is of most interest in the interval of parameter variation containing bifurcation values. After the completion of the parameter change such systems come to a distinct final state. If there are several possible states, the system has to choose one of them. For example, after the first period-doubling bifurcation, a doubled period motion can be realized by two ways differing only by the phase shift [Bezruchko *et al.*, 2003].

In the case of very slow (adiabatic) change of the parameters even a small noise in the system exerts a crucial influence on the choice of this or that final state. If the noise distribution is symmetric with respect to the final states, then these states are established with the same probability p. This is the case of the so-called probability symmetry [Brush *et al.*, 1995]. For instance, if there are only two possible final states, then $p_1 = p_2 = 1/2$. In the case of infinitesimal noise either the first ($p_1 = 1$, $p_2 = 0$) or the second ($p_1 = 0$, $p_2 = 1$) of these two final states is selected, depending on the initial conditions, with a unit probability that corresponds to completely predictable behavior. These limiting cases of the bifurcation transition are called the "stochastic" and "dynamic" variants, respectively [Brush *et al.*, 1995]. In real systems, certain intermediate situations are realized in which both the noise and the rate of the control parameter variation are significant. Nevertheless, a conditional boundary between stochastic and dynamic transitions can be defined using a criterion according to which the possible final state is attained at a preset probability (e.g. $p_1 = 0.75$, $p_2 = 0.25$).

Previously, the phenomenon of violation of the probability symmetry of the final states in a system with time-varying parameters in the presence of noise was numerically studied by Butkovskii *et al.*

[1996], Bezruchko and Ivanov [2000], and Bilchinskaya *et al.* [2002] using one-dimensional maps of the type $x_{n+1} = f(x_n + \xi_n, r)$, where f is a quadratic function, ξ_n is the noise possessing zero mean and symmetric distribution, and r is the bifurcation parameter varying in time according to a piecewise-linear law. The aim of the present paper is to examine the violation of the final states probability symmetry for an experimental system with continuous time, and for a coupled system.

2. Experimental Study of Fast Bifurcation Transitions

The experimental study of fast bifurcation transitions was carried out using an electronic oscillator with delayed feedback. A block diagram of the experimental setup is shown in Fig. 1. The delay line and the nonlinear element were constructed using digital elements. They were connected with the analogue low-frequency RC-filter with the help of analog-to-digital and digital-to-analog converters. The noise signal from a controlled noise generator was entered into the ring oscillator using a summing amplifier. The characteristic of the nonlinear element had the form $f(x) = r - x^2$, where the control parameter r was varied in time as

$$r(t) = \begin{cases} r_1 + st, & t \le T, \\ r_1 + sT = r_2, & t > T. \end{cases} \quad (1)$$

Here r_1 and r_2 are the initial and final values of the parameter r variation, s is the rate of the control parameter variation determined as $s = (r_2 - r_1)/T$, and T is the time within which the parameter r changes from r_1 to r_2 coming through the first period-doubling bifurcation value r_B.

For different values of the noise variance σ^2 we determine the probability p_1 of one of the two possible final states of the system, Fig. 2. The final states differing from each other by the phase of the period-2 oscillations were defined from the voltage time series for different s values. In order to accumulate sufficient statistics, necessary for calculating mean probabilities, 2000 experiments were carried out with each set of the noise variance and the parameter variation rate. If the rate of the control parameter variation is very small, then the time within which the parameter changes by an appreciable value is much greater than the characteristic time of the system relaxation. As a result, after the period-doubling bifurcation the system stays in the vicinity of the cycle having lost the stability. In this case, the choice of the final state is defined mainly by noise. If the noise distribution is symmetric with respect to the final states, then these states are established with the same probability even for a relatively low level of noise. As the parameter variation rate grows, the results of the bifurcation transition become more predictable. The increasing rate of the control parameter change leads to the motion of boundaries of basins of attraction of final states in the phase space. The choice of the initial conditions begins to exert more influence on the choice of the final state, Fig. 2.

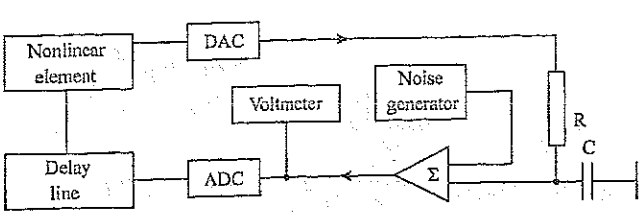

Fig. 1. Block diagram of the electronic oscillator with delayed feedback. ADC is the analog-to-digital converter, DAC is the digital-to-analog converter, and Σ is the summing amplifier.

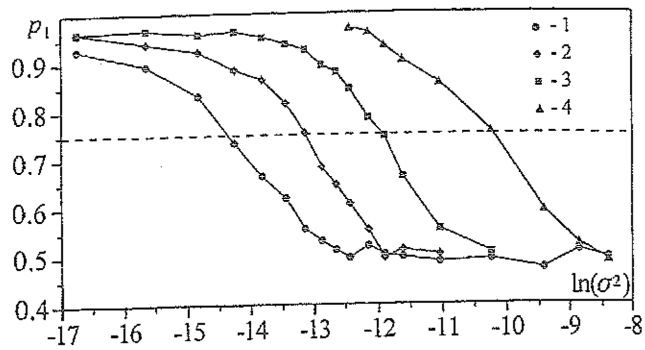

Fig. 2. Probability p_1 of a chosen final state versus noise intensity logarithm $\ln(\sigma^2)$ for the electronic oscillator with delayed feedback. The curve numbers correspond to $s = 6.75 \times 10^{-3}$ (1), $s = 13.5 \times 10^{-3}$ (2), $s = 27 \times 10^{-3}$ (3), and $s = 67.5 \times 10^{-3}$ (4).

As the noise variance increases for a given value of s, the probability p_1 decreases unless it becomes close to 0.5. With further increasing of noise the mean value of probability p_1 fluctuates in the vicinity of 0.5 level, Fig. 2. In this case the mean values of p_1 are close to each other for different s values. Some points of the curves 2 and 3 are not shown in this region in order to avoid overloading of the figure. For the same reason, the points of the curve 4 are not shown for small levels of noise corresponding to the case where the mean values of probability p_1 are greater than 0.97. Each point in Fig. 2 being a mean value has its own error bars. For every point in the plot the error of p_1 definition does not exceed 3%.

The noise intensity varies in more than 1000 times for the points of the curves in Fig. 2 and thus, the region of comparison is sufficiently wide.

The obtained results agree well with the results of numerical simulation. In particular, the dependence of probability of a chosen final state on the noise intensity observed in the experimental system is qualitatively similar to that observed for the quadratic maps [Butkovskii *et al.*, 1996; Bezruchko & Ivanov, 2000; Bilchinskaya *et al.*, 2002].

3. Basins of Attraction of Final States for a System of Coupled Maps with Varying Parameters

To investigate the violation of the final states probability symmetry in coupled systems with time-varying parameters, we consider a system of two coupled quadratic maps with additive noise

$$\begin{cases} x_{n+1} = r_n - (1-k)(x_n + g_n)^2 - k y_n^2, \\ y_{n+1} = r_n - (1-k)(y_n + h_n)^2 - k x_n^2, \end{cases} \tag{2}$$

where x, y are the dynamical variables, k is the coupling coefficient, r_n is the control parameter, g_n and h_n are the noise processes. The parameter r_n depends on time by a piecewise-linear law

$$r_n = \begin{cases} r_{n-1} + s, & 1 \le n < N, \\ r_F = \text{const}, & n \ge N, \end{cases} \tag{3}$$

where s is the rate of the control parameter variation, r_0 and r_F are its initial and final values, respectively, N is the number of steps used to pass the interval (r_0, r_F).

We examine the case of $r_0 = 0.5$ and $r_F = 1$, where the interval of the control parameter variation contains only one bifurcation value $r_B = 0.75$ corresponding to the first period-doubling bifurcation of a single map. To finish the transients, 1000 iterations were executed at the fixed parameter r_F. Additive noise g_n was uniformly distributed in the interval $(-\gamma_g, +\gamma_g)$ and was assumed to be uncorrelated: $\langle g_n g_m \rangle = \sigma_g^2 \delta_{mn}$, where $\sigma_g^2 = \gamma_g^2/3$ is the variance, δ_{mn} is the Kronecker delta. The noise process h_n has similar features. The values of g_n and h_n are assumed to be independent: $\langle g_n h_m \rangle = 0$.

For $r_n = r_F$ the isolated quadratic map has two equivalent final states: $x_1 = \{x^+, x^-, x^+, x^-, \dots\}$ and $x_2 = \{x^-, x^+, x^-, x^+, \dots\}$, where x^+ and x^- are the greater and the smaller values, respectively, of the dynamical variable x_n for the period-2 cycle. The coupled map system (2) has four equivalent final states for $k = 0$ and $r_n = r_F$: $z_{11} = (x_1, y_1)$, $z_{12} = (x_1, y_2)$, $z_{21} = (x_2, y_1)$, and $z_{22} = (x_2, y_2)$, where z_{11} and z_{22} are the in-phase states ($x_1 = y_1$, $x_2 = y_2$), and z_{12} and z_{21} are the out-of-phase ones. At nonzero k, depending on its value, one or two pairs of final states are possible that correspond to motions on the in-phase and the out-of-phase cycles of the coupled system. The choice of these states depends on initial conditions (x_0, y_0), noise level, the rate s of the control parameter variation, and the coupling coefficient k.

First, we consider the configuration of final state basins of attraction for period-2 cycles of system (2) depending on the parameters k and s in the absence of noise. In Fig. 3 the basins are shown for the case $k = 0$. If the initial point (x_0, y_0) is beyond the basin of attraction, the solution goes to infinity (this region is shown in white in Fig. 3 and in the subsequent figures with the basins). The points of the attractors are indicated by circles (in-phase cycle) and squares (out-of-phase cycle) in white or black depending on background. For dynamical regimes the basin structure qualitatively differs from that for the stationary case. The region of finite solutions decreases and the basins vary in size as the result of redistribution of various final states probability depending on the rate of the bifurcation parameter variation. For example, for the pair z_{11} and z_{22} of final states and the pair z_{12} and z_{21} the inversion of basins is observed depending on the parity or oddness of the step number N. It is interesting to note that, if we prescribe the initial conditions on one of the attractors, the system with varying parameters can evolve to the final state corresponding to another attractor.

For $k \neq 0$ some final states observed in the stationary case at r_F are not observed in the system with time-varying parameters unless the rate of the control parameter variation is below some critical value. For instance, for $0 < k < 1$ the interval of the parameter r is observed within which only the in-phase attractor exists in the stationary system after period-doubling bifurcation, Fig. 4. In the dynamical case at a very small rate of the control parameter variation in this interval, the system has time to settle on the in-phase attractor for any initial conditions. As a result, the entire region of finite solutions becomes the basin of attraction of the in-phase regime and its respective final states. With the parameter s increasing there appears narrow fragments of basins of the out-of-phase attractor final states which expand for large s, Fig. 5.

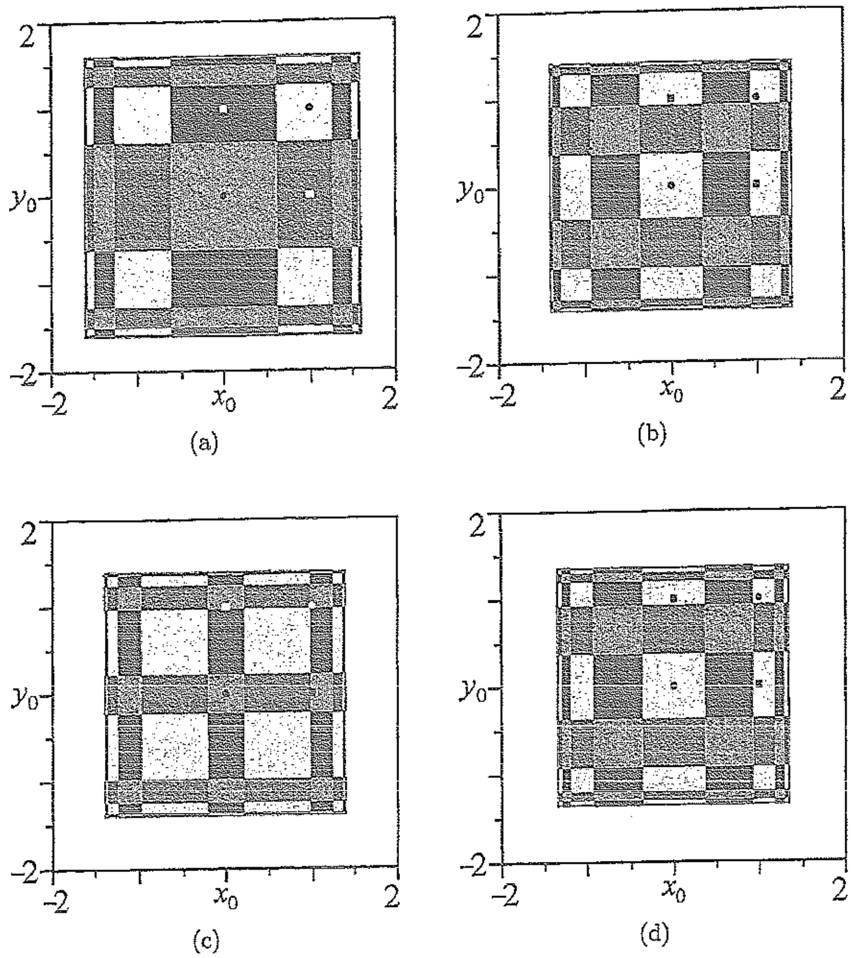

Fig. 3. Basin structure of the final states z_{11} (light gray), z_{22} (gray), z_{12} (dark gray), and z_{21} (black) in the absence of noise ($g_n = h_n = 0$) and $k = 0$. (a) The control parameter is fixed, $r_n = 1$, (b) $s = 0.25$, (c) $s = 0.167$, (d) $s = 0.01$.

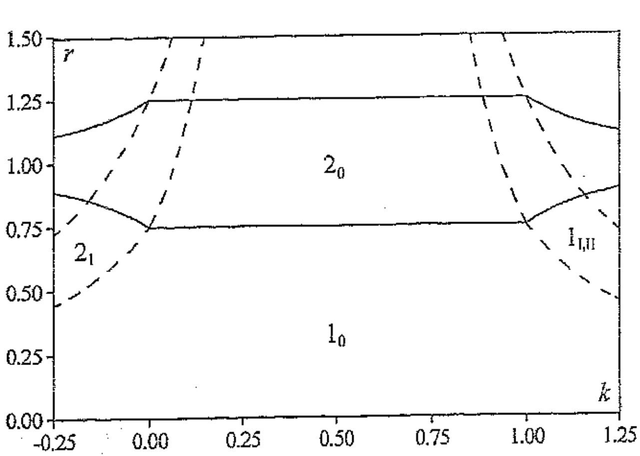

Fig. 4. Stability regions for the period-1 and period-2 cycles of the system (2) in the (k, r) plane for the case of the parameters not varying in time and $g_n = h_n = 0$. 1_0 and 2_0 are the in-phase period-1 and period-2 cycles, respectively. $1_{I,II}$ and 2_I are the out-of-phase cycles of periods 1 and 2, respectively. Solid lines are the lines of period-doubling bifurcations. The regions of stability of out-of-phase cycles are bounded with dashed lines.

Similarly to the stationary case, the basins of the final states z_{11} and z_{22} corresponding to the in-phase attractor become convex at dynamical bifurcations and the basins of the out-of-phase attractor final states z_{12} and z_{21} become concave as the coupling coefficient increases. For some k values the out-of-phase period-2 cycle loses stability and basins of attraction of z_{12} and z_{21} states disappear. For $k = 0.5$ the basins of z_{11} and z_{22} final states, which have been rectangular for zero k, take the form of concentric circles. The region of finite solutions is maximal in this case.

In the region of large k values the coupled system also demonstrates the out-of-phase regimes. If the system (2) parameters do not vary in time and noise is absent, the bifurcation lines of out-of-phase regimes are symmetric with respect to the line $k = 0.5$, Fig. 4. However, the oscillation states at strong coupling differ radically from those at weak coupling [Bezruchko et al., 2003]. For instance,

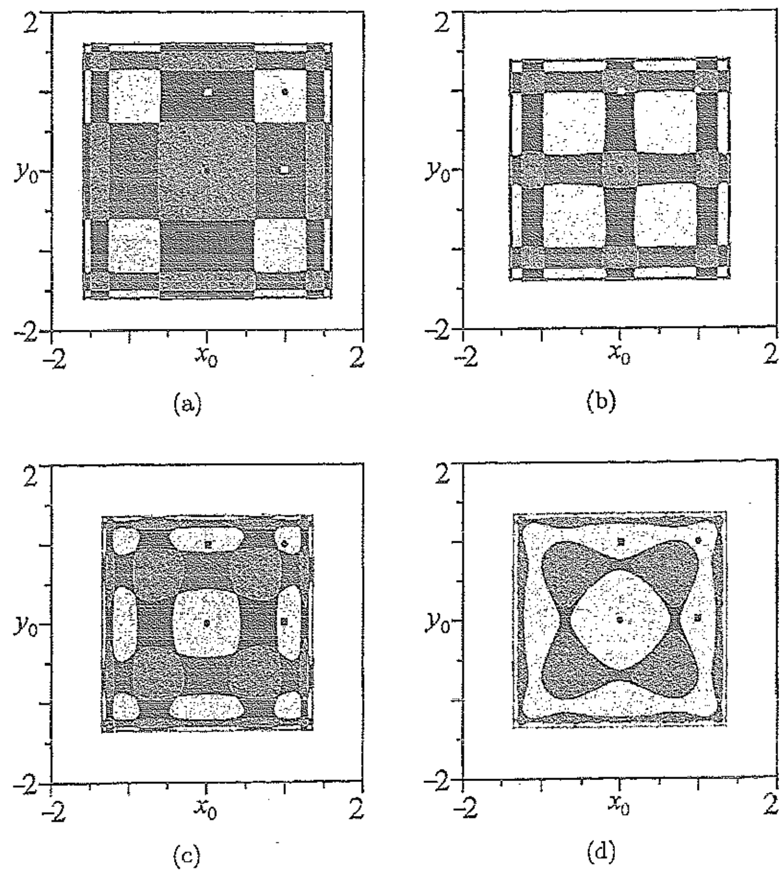

Fig. 5. Basins of the final states z_{11}, z_{22}, z_{12}, and z_{21} for $k = 0.005$ in the absence of noise. (a) The control parameter is fixed, $r_n = 1$, (b) $s = 0.167$, (c) $s = 0.01$, (d) $s = 0.002$.

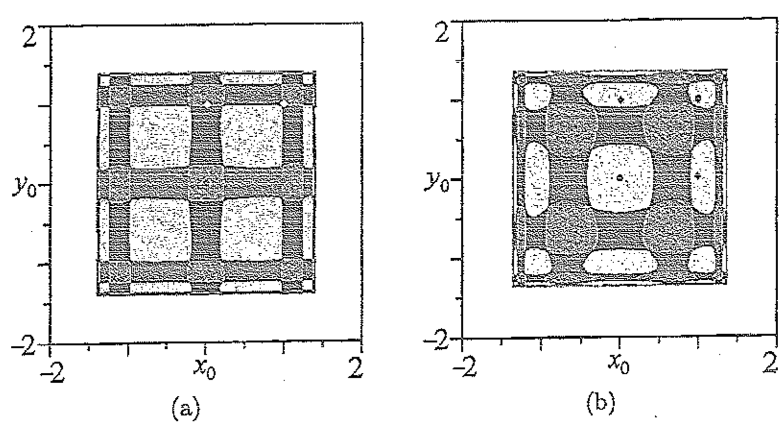

Fig. 6. Basins of the final states z_{11} (light gray), z_{22} (gray), z^{12} (dark gray), and z^{21} (black) in the absence of noise and $k = 0.995$. (a) $s = 0.167$, (b) $s = 0.01$.

in the stationary case two out-of-phase period-1 cycles, denoted as 1_{I} and 1_{II}, exist in the system for $k \geq 2/3$. The stability region of these cycles is symmetric to the stability region of the out-of-phase period-2 cycle with respect to the line $k = 0.5$. The final states corresponding to the cycles 1_{I} and 1_{II}, denoted by z^{12} and z^{21}, coincide with the out-of-phase period-2 cycle final states z_{12} and z_{21}, but in distinction to them, z^{12} and z^{21} do not transfer one into another at successive iterations. As a result, for strong coupling in dynamical regime there is no inversion of out-of-phase period-1 states basins depending on the parity or oddness of N, Fig. 6. The other feature of final state basins at $k > 0.5$ are similar to those observed in the case of $k < 0.5$.

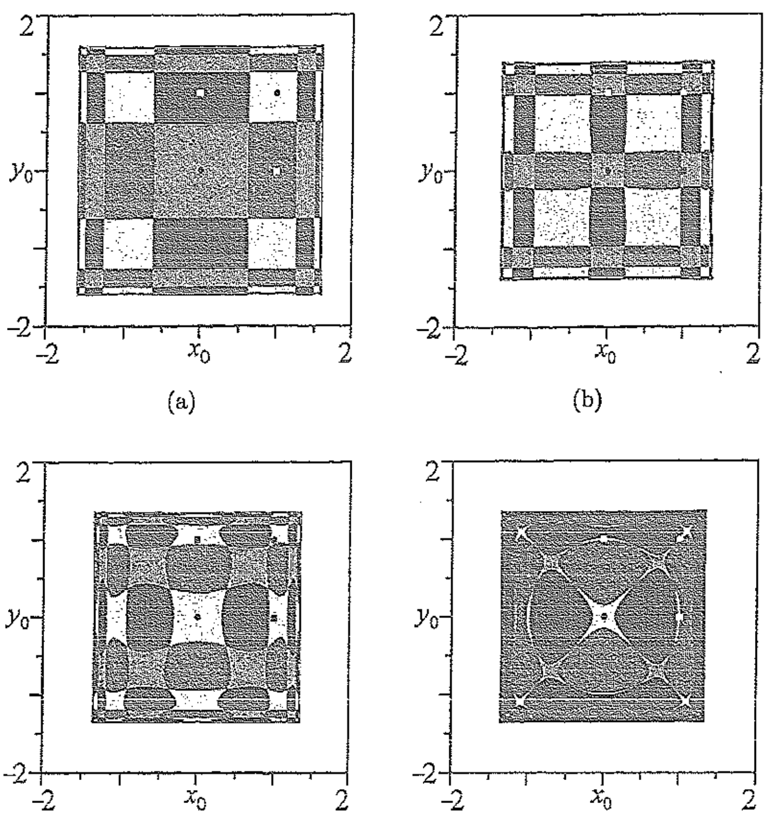

Fig. 7. Basins of the final states z_{11}, z_{22}, z_{12}, and z_{21} for $k = -0.005$ in the absence of noise. (a) The control parameter is fixed, $r_n = 1$, (b) $s = 0.167$, (c) $s = 0.01$, (d) $s = 0.002$.

For coupling values $k < 0$ and $k > 1$ the out-of-phase period-2 and period-1 cycles, respectively, emerge at smaller values of the control parameter r than the in-phase period-2 cycle does, Fig. 4. Consequently, the interval of the parameter r is observed within which only the out-of-phase attractors exist in the coupled system. This fact results in the preference of the out-of-phase regimes at small rates of the control parameter variation. The probability of their final states increases at dynamical bifurcations at the expense of the in-phase regimes probability. The redistribution of basins area is illustrated in Fig. 7. Thus, depending on the coupling between elements and the rate of their parameters variation one can obtain the required final state of the system and thereby control its dynamics.

To illustrate the probability of the system (2) possible states we calculated the relative area A of every final state basin against the coupling coefficient, Fig. 8. By virtue of the final states (x_1, y_2) and (x_2, y_1) specular symmetry the areas of their basins coincide for any k and s values, and the curves for z_{11} and z_{22} final states exchange places at every iteration step.

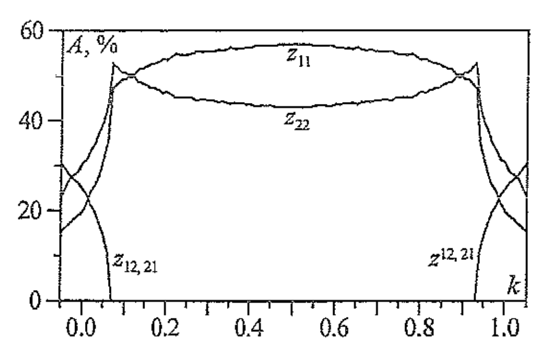

Fig. 8. Dependence of the relative area A of the final state basins on the coupling coefficient for $s = 0$, $g_n = h_n = 0$.

The noise adding leads to smearing of the clearly defined boundaries of basins and to equalization of probability of all final states. The greater is the noise level and the smaller is the rate of the control parameter variation, the greater is the effect of probability equalization, Fig. 9. For other k values the effect of noise on the basin structure is qualitatively similar to that presented in Fig. 9. For other intensities of noise we obtain the plots qualitatively similar to the presented ones but with other values of s. Besides the case of uniformly distributed noise,

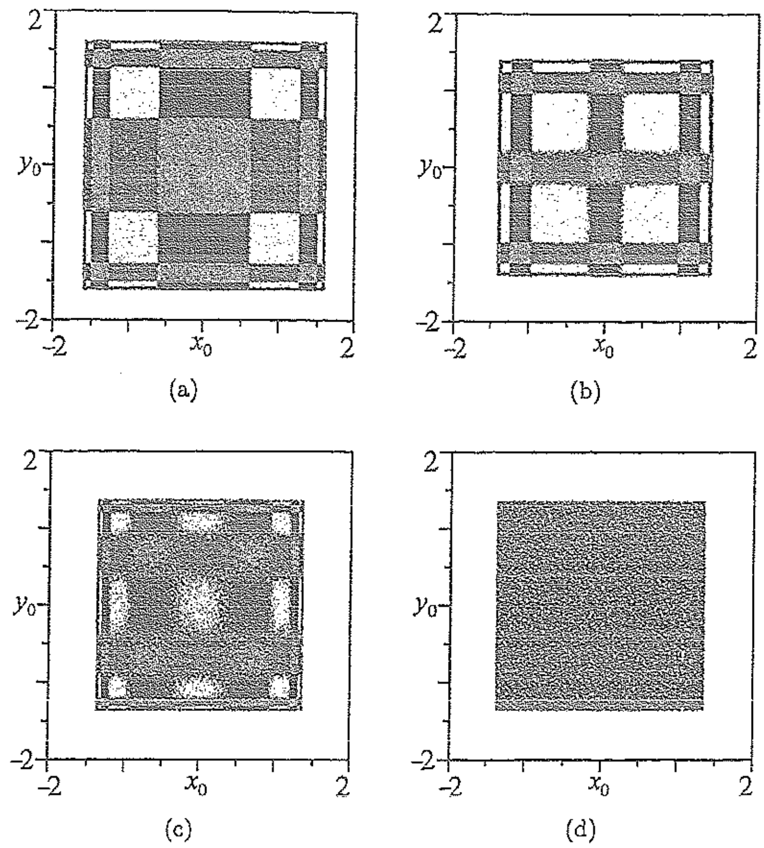

Fig. 9. Basin structure of the final states z_{11}, z_{22}, z_{12}, and z_{21} for $k = 0.005$ in the presence of noise, $\sigma_g^2 \approx \sigma_h^2 \approx 10^{-4}$. (a) The control parameter is fixed, $r_n = 1$, (b) $s = 0.167$, (c) $s = 0.025$, (d) $s = 0.01$.

we also considered the case of Gaussian noise which distribution was symmetric with respect to the final states. The obtained results were qualitatively similar for both types of noise.

4. Conclusion

The obtained results extend the notion about the features of fast bifurcations in dynamical systems and systems with noise. The violation of the final states probability symmetry is observed for the experimental system with continuous time and for the coupled map system. It is shown that the selection of final states in these systems depends on the rate of the control parameter variation and noise level. The existence of coupling between elements enhances the effect of probability symmetry break inherent in uncoupled elements for small noise level. In the presence of noise the equalization of probabilities of all possible final states takes place.

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