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# Recovering Parameters of Time-Delay Systems from Transient Time Series

**V. I. Ponomarenko and M. D. Prokhorov***

*Institute of Radio Engineering and Electronics (Saratov Branch), Russian Academy of Sciences, Saratov, Russia*

**e-mail: sbire@sgu.ru*

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**Abstract**—A method is proposed for the recovery of parameters of delayed-feedback systems performing periodic oscillations. The proposed method is based on an analysis of the system response to a regular external action that leads to the onset of a transient process.

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In recent years, the problem of determining the parameters of systems with delayed feedback (time-delay systems) from the corresponding time series of observable dynamic variables has received much attention. Various methods have been proposed that make it possible to recover the model equations of time-delay systems using the chaotic time series of their oscillations [1–9]. Unfortunately, these methods are not effective in cases where such systems occur in periodic regimes [10]. However, many of the practically important dynamical time-delay systems can operate in periodic or nearly-periodic regimes [11, 12]. Therefore, the task of developing methods using time series for the recovery of parameters of time-delay systems performing periodic oscillations is of considerable importance. First attempts in this direction were recently undertaken by Siefert [13], who proposed to estimate the delay by monitoring the system response to an external noise (with a large amplitude and a correlation time much shorter than the delay time of the system under consideration) and analyze the corresponding correlation function.

This Letter describes an alternative approach to the recovery of parameters of time-delay systems in a periodic regime. The proposed method is based on an analysis of the system response to a regular external action that leads to the onset of a transient process in the system under consideration.

Let us consider a system described by the following first-order differential equation with a delayed argument:

$$\varepsilon_1 \dot{x}(t) = -x(t) + f(x(t-\tau_1)), \tag{1}$$

where $x$ is the dynamical variable, $\tau_1$ is the delay time, $\varepsilon_1$ is a parameter characterizing the inertia of the system, and $f$ is a nonlinear function. Previously, it was established [8] that time series of the time-delay systems of type (1) contain virtually no extrema spaced from each other by $\tau_1$. If system (1) performs chaotic oscillations, the extrema in the time series are irregularly encountered and spaced by various time intervals. Taking into account this behavior, we proposed a method for determining $\tau_1$ using statistical analysis of the time intervals between extrema in the chaotic time series of a given time-delay system. According to the proposed procedure, the numbers $N$ of situations whereby the pairs of points spaced by $\tau$ simultaneously represent the extrema in the given time series are determined for various $\tau$ and the $N(\tau)$ plot is constructed. Then, $\tau_1$ is readily determined as the value corresponding to the absolute minimum in this plot [8].

However, this method fails to operate if system (1) performs periodic self-sustained oscillations and, hence, the extrema in the time series are encountered in a regular manner. As a result, the $N(\tau)$ plot exhibits several peaks separated by the intervals where $N = 0$. Figure 1 shows an example of such a time series and the corresponding $N(\tau)$ plot for system (1) with $\tau_1 = 300$, $\varepsilon_1 = 10$, $f(x) = \lambda - x^2$, and $\lambda = 1.0$. For the given nonlinearity parameter $\lambda$, system (1) exhibits periodic motions with a period of $T_a = 619$ (Fig. 1a). Since the time series is asymmetric, the $N(\tau)$ plot (Fig. 1b) displays two peaks (at $\tau = 305$ and $\tau = 314$), which correspond to the distances between maxima $B$ and minima $C$ and between minima $C$ and maxima $D$, respectively. Thus, the $N(\tau)$ plot cannot be used for determining the delay from the time series of periodic oscillations in system (1).

Let an external signal $F(t)$ be applied to system (1), which is now described by the following equation:

$$\varepsilon_1 \dot{x}(t) = -x(t) + f(x(t-\tau_1)) + F(t). \tag{2}$$

Consider an external signal $F(t)$ representing rectangular pulses with amplitude $A$, period $T$, and duration $M$.

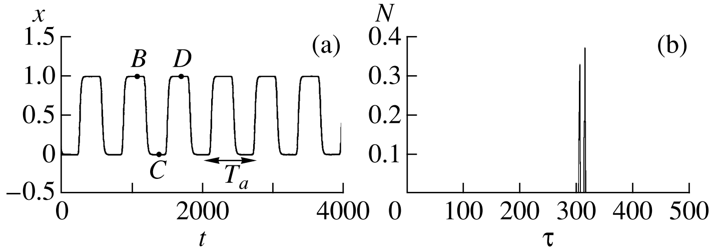


**Fig. 1.** Time-delay system (1) performing oscillations in a periodic regime: (a) a typical time series of oscillations in the system; (b) a plot of the number $N$ of the pairs of extrema in the given time series spaced by various time intervals $\tau$ (normalized to the total number of extrema).

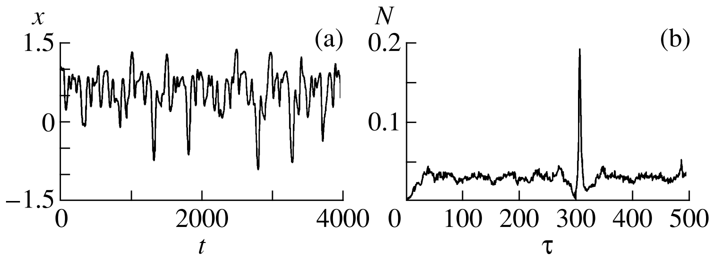


**Fig. 2.** Reconstruction of the delay time of system (2): (a) typical time series of oscillations in the system; (b) a plot of the number $N$ of the pairs of extrema in the given time series spaced by various time intervals $\tau$ (normalized to the total number of extrema), which shows that $N_{\min}(\tau) = N(300)$.

If this action is strong enough, this will lead to the onset of a transient process. As a result, the system performs motions in a broader region of the phase space, which provides additional information on the system dynamics that helps to recover its parameters [14]. In particular, the appearance of a large number of additional extrema in the regions of time series corresponding to the transient process leads to the appearance of a pronounced minimum on the $N(\tau)$ plot, which can be used to determine the delay time.

Figure 2 shows a time series and the corresponding $N(\tau)$ plot for system (2) with the same parameters as those for the autonomous system (1) (illustrated in Fig. 1) and the external signal parameters $A = 0.5$, $T = 490$, and $M = 0.2T$. As can be seen, the time series is similar to a chaotic one (Fig. 2a) and contains a large number of irregular extrema. This makes possible to estimate exactly the delay from the $N(\tau)$ plot constructed at a unit step in $\tau$, which reveals a clear minimum at $\tau = \tau_1 = 300$ (Fig. 2b).

Investigations show that the proposed method of determining the delay time is applicable in a broad range of parameters of the external action. For $A = 0.5$ and $M = 0.2T$, the repetition period $T$ of external pulses, for which the $\tau_1$ value is exactly determined, can be arbitrarily selected in the interval from $T = 1.2\tau_1$ to $1.8\tau_1$. There are narrower intervals of $T$ (below $\tau_1$ and above $2\tau_1$) where the delay time is also accurately estimated. It should be noted that, in selecting $T$, it is possible to use a rough estimate for the delay time of an autonomous system (1), according to which $\tau_1$ is always smaller than $T_a/2$. We have also verified the proposed procedure by varying the external pulse duration within broad limits (from $M = 0.05T$ to $0.5T$). The method remained effective, but small $M$ values make necessary an increase in the pulse amplitude $A$ (which can be decreased with increasing $M$). It should be noted that the delay can also be estimated using a harmonic external action $F(t)$, provided that its parameters are such that the initial time-delay system exhibits chaotic behavior under this action.

Once $\tau_1$ is determined, the inertial parameter $\varepsilon_1$ can be recovered and the nonlinear function $f$ reconstructed from the unperturbed periodic time series (Fig. 1a) using an algorithm described previously [8] (in application to the chaotic time series of a time-delay system). In order to implement this approach, the points of a given time series are plotted on the plane of coordinates $[x(t-\tau_1), \varepsilon\dot{x}(t) + x(t)]$, the parameter $\varepsilon$ is varied at a certain step, and the length of a straight segment $L(\varepsilon)$ connecting points (ordered with respect to abscissa $x(t-\tau_1)$ in this plane is determined. If $\varepsilon = \varepsilon_1$, then, according to Eq. (1), the points on the indicated plane reproduce the nonlinear function and the length $L(\varepsilon)$ of the connecting segment exhibits a minimum. The greater the error of estimated parameters, the less ordered are the points and the longer is the broken line connecting these points as compared to the case where these points fit to a one-dimensional curve.

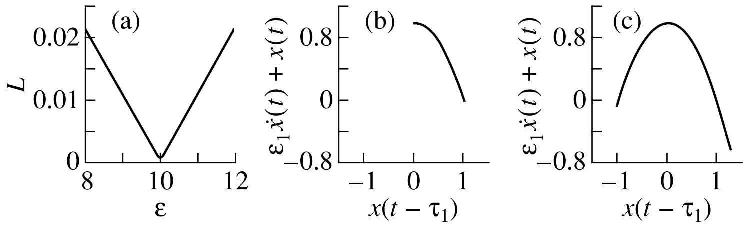


**Fig. 3.** Reconstruction of the inertial parameter ε and nonlinear function $f$ upon the recovery of delay time $\tau_1$: (a) a plot of the length $L$ of the straight segment connecting points (ordered with respect to abscissa) on the $[x(t-\tau_1), \varepsilon\dot{x}(t) + x(t)]$ plane for ε varied at a certain step; (b, c) nonlinear functions $f$ reconstructed from the periodic time series of systems (1) and (2), respectively.

Figure 3a shows the $L(\varepsilon)$ curve constructed with ε varied at a step of 0.01 for the delay time $\tau_1 = 300$ recovered as described above. As can be seen, the $L(\varepsilon)$ function has a minimum at $\varepsilon = \varepsilon_1 = 10.00$. Figure 3b presents the nonlinear function reconstructed for $\tau_1 = 300$ and $\varepsilon_1 = 10.00$ from a periodic time series of system (1). Note that, using this approach, only a fragment of this function can be reconstructed because, by virtue of the regular character of motions, the oscillations are performed in a small region of the phase space. For a more complete recovery of the nonlinear function, it is possible to use a time series of the nonautonomous system (2). In this case, the plot of $\varepsilon_1\dot{x}(t) + x(t)$ versus $x(t-\tau_1)$ should be constructed using only points of the time series corresponding to the intrinsic dynamics of the time-delay system (i.e., points should be selected in the intervals between sequential pulses of the external action). Figure 3c shows the nonlinear function reconstructed in this way, which quite well coincides with the true quadratic function of system (1).

The proposed method also remains effective in the presence of a noise. In order to check for this, we applied the procedures described above to a time series obtained by adding a Gaussian white noise with zero mean to the initial time series corresponding to Eq. (2). In the case where the added disturbance had a standard deviation amounting to 10% of that for the time series without noise, the position of $N(\tau)$ minimum still allowed the delay to be exactly recovered. The values of $\varepsilon_1$ and the nonlinear function were also recovered with a good precision, but it was necessary to employ the time series of a non autonomous system and use only points selected in the intervals between sequential pulses of the external action.

In conclusion, we proposed a method for the recovery of delay time, inertial parameter, and nonlinear function of a time-delay system performing periodic oscillations. The proposed method is based on an analysis of the system response to a regular external action that leads to the onset of a transient process, which provides additional information on the system dynamics and helps recovering the required parameters.

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