ISSN 1063-7850, Technical Physics Letters, 2008, Vol. 34, No. 4, pp. 331–334. © Pleiades Publishing, Ltd., 2008.
Original Russian Text © M.D. Prokhorov, V.I. Ponomarenko, 2008, published in Pis'ma v Zhurnal Tekhnicheskoĭ Fiziki, 2008, Vol. 34, No. 8, pp. 29–35.

Reconstructing Model Equations of the Chains of Coupled Delay-Feedback Systems from Time Series

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Revised manuscript received October 8, 2007

Abstract—A method for reconstructing model delay-differential equations for the chains of coupled delay-feedback systems from their time series is described for the first time. The efficacy of the proposed method is demonstrated by an analysis of the chaotic time series for the chains of both unidirectionally and mutually coupled time-delay systems described by the Ikeda equations and the Mackay–Glass equations with high noise levels.

PACS numbers: 05.45.Tp
DOI: 10.1134/S1063785008040184

Systems with delayed feedback (time-delay systems) are widely encountered in both nature and technology [1]. In contrast to the investigation of dynamics of the autooscillatory time-delay systems, the problem of reconstructing the model equations for such systems from their time series is studied in less detail. Since even simple systems with delayed feedback can exhibit chaotic oscillations of very high dimensionality, universal methods for their reconstruction are ineffective and, hence, special approaches have to be developed for the recovery of equations of particular delay-feedback systems [2–8]. However, all approaches developed previously were aimed at the recovery of isolated time-delay systems, whereas situations involving several interacting systems with delayed feedback are typically encountered in solving important applied problems [9–11].

Recently, we have proposed [12] a method for the recovery of two linearly coupled time-delay system from their chaotic time series. This Letter presents the development of our method for the case of an arbitrary number of interrelated delay-feedback systems and describes a method for the reconstruction of model delay-differential equations from the time series of the chains of coupled delay-feedback systems with various types of coupling between elements of the chain.

Let us first consider a chain consisting of three unidirectionally coupled time-delay systems described by the following system of differential equations:

$$\varepsilon_1\dot{x}_1(t) = -x_1(t) + f_1(x_1(t-\tau_1)), \tag{1}$$

$$\varepsilon_2\dot{x}_2(t) = -x_2(t) + f_2(x_2(t-\tau_2)) + k_1[x_1(t) - x_2(t)], \tag{2}$$

$$\varepsilon_3\dot{x}_3(t) = -x_3(t) + f_3(x_3(t-\tau_3)) + k_2[x_2(t) - x_3(t)], \tag{3}$$

where τ_1, τ_2, and τ_3 are the delay times; ε_1, ε_2, and ε_3 are parameters characterizing the inertial properties of the systems; f_1, f_2, and f_3 are nonlinear functions; and k_1 and k_2 are the coupling coefficients. This type of coupling between time-delay systems and the resulting synchronization of oscillations in elements of the chain have been previously studied in [9, 13].

For recovering the delay times τ_i from the time series $x_i(t)$, let us use the method proposed previously [8], which is based on the fact that time series of the delay-feedback systems of type (1) contain virtually no extrema separated from one another by an interval equal to the delay time. Then, in order to recover τ_i, it is necessary to find extrema in a given time series, determine the number N of the pairs of extrema spaced from each other by various intervals τ in this time series, construct the $N(\tau)$ plot, and estimate the delay time τ_i as the position of the absolute minimum of this function. Investigations showed that this method of determining the delay times can be also successfully applied to coupled time-delay systems, provided that their interaction does not lead to the appearance of a large umber of additional extrema in the time series.

System (1) is not subjected to the action of other systems and, hence, the parameter ε_1 and the nonlinear function f_1 can be recovered using the aforementioned method [8]. In order to reconstruct the model equations (2) and (3) of the second and third systems in the chain, we propose to use the following approach. As can be seen from Eqs. (2) and (3), the dependences

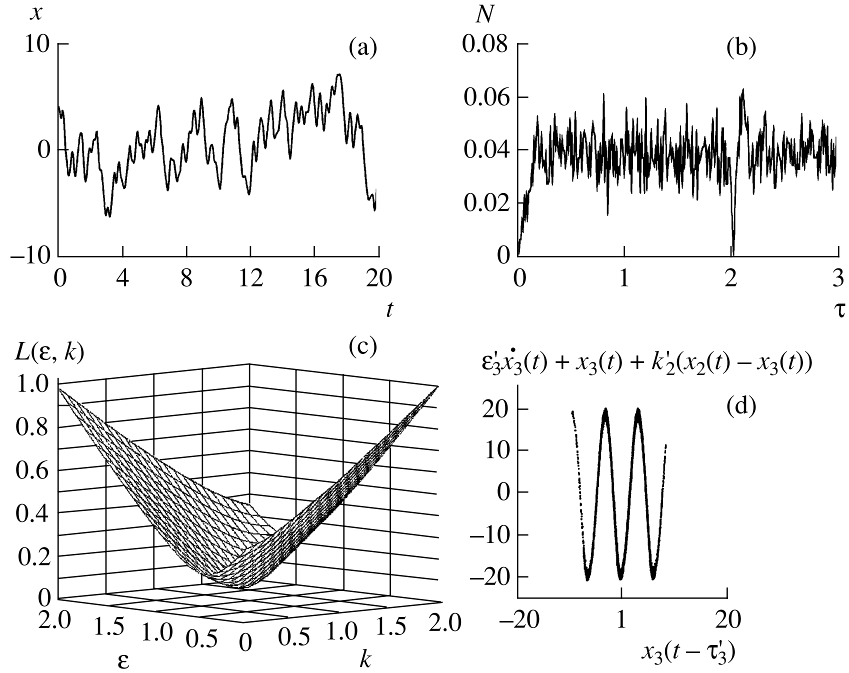

Fig. 1. Reconstruction of an element of a chain of unidirectionally coupled Ikeda systems (1)–(3) in the presence of a 20% white Gaussian noise: (a) the typical experimental time series of $x_3(t)$; (b) a plot of the number N of the pairs of extrema spaced by various distances τ in the given time series (normalized to the total number of extrema), which yields $N_{min}(t) = N(2.00)$); (c) a plot of $L(\varepsilon, k)$ (normalized to the maximum value $L_{max}(\varepsilon, k) = L(0.0, 2.0)$ in a given interval of parameters, which yields $L_{min}(\varepsilon, k) = L(0.98, 0.95)$); (d) nonlinear function f_3 reconstructed for $\tau'_3 = 2.00$, $\varepsilon'_3 = 0.98$, $k'_2 = 0.95$.

of $\varepsilon_{2,3}\dot{x}_{2,3} + x_{2,3}(t) - k_{1,2}(x_{1,2}(t) - x_{2,3}(t))$ on $x_{2,3}(t - \tau_{2,3})$ reproduces the nonlinear functions f_2 and f_3, respectively. In order to determine the unknown parameters $\varepsilon_{2,3}$ and $k_{1,2}$, we suggest to try various values so as to obtain single-valued relationships on the corresponding planes, which is only possible provided a correct choice of these parameters. As a quantitative criterion of such a unique relationship in the search for $\varepsilon_{2,3}$ and $k_{1,2}$, we can use the minimum length $L(\varepsilon, k)$ of a broken line connecting sequential points (ordered with respect to the abscissa) on the corresponding planes. If the choice of $\varepsilon_{2,3}$ and $k_{1,2}$ is incorrect, we will obtain a set of points that do not obey a functional relationship. The lower the accuracy in the choice of parameters, the less ordered are the points and the longer is the length of the broken line connecting these points (compared to the case when this set of points belongs to a one-dimensional curve). In order to reduce the computational time, the step of variation of the ε and k values can be initial taken relatively large and then decreased in the vicinity of the minimum of $L(\varepsilon, k)$.

As an application example, Fig. 1 illustrates the recovery of Eq. (3) from very noisy chaotic time series of the variables $x_2(t)$ and $x_3(t)$ in the chain of unidirectionally coupled Ikeda systems [14] described by the following equations:

$$\dot{x}_i(t) = -x_i(t) + \mu_i \sin(x_i(t-\tau_i) - x_{0i}) + k_{i-1}[x_{i-1}(t) - x_i(t)], \quad (4)$$

where $i = 1, 2, 3$ and we use the boundary condition $x_0 \equiv x_1$, which implies that the first element is not subjected to the action of other elements. The parameters of all three Ikeda equations were set identical ($\mu_{1,2,3} = 20$, $\tau_{1,2,3} = 2$, $x_{01,02,03} = \pi/3$), while the initial conditions were different. Systems with such parameters demonstrate the motion on a chaotic attractor of high dimensionality [14]. Equations (4) correspond to $\varepsilon_i = 1$. The coefficients of coupling between elements in the chain were different: $k_1 = 1.5$, $k_2 = 1$. It should be noted that synchronization of the unidirectionally coupled Ikeda systems with the indicated parameters is observed for $k_1 = k_2 \geq 7.5$. All three coupled systems were rendered noisy by adding white Gaussian noise with a zero mean and a mean-square deviation amounting to 20% of that for the noiseless time series (which corresponds to a signal-to-noise ratio about 14 dB).

Figure 1a shows a fragment of the time series of oscillations in the third element of the chain under consideration. The scale was such that a temporal interval equal to the delay time accommodated 200 points of the time series. Despite a high noise level, the

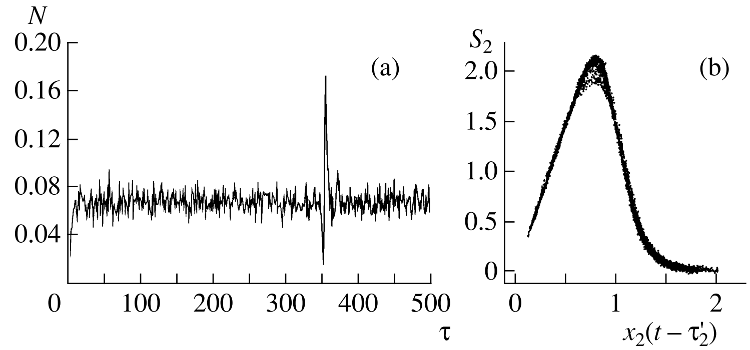

Fig. 2. Reconstruction of an element of a chain of diffusion-coupled Mackay–Glass systems (6) in the presence of a 10% white Gaussian noise: (a) a plot of the number N of the pairs of extrema spaced by various distances τ in the time series of $x_2(t)$ (normalized to the number of points in time series), which yields $N_{min}(\tau) = N(350)$; (b) nonlinear function f_2 reconstructed for $\tau_2' = 350$, $\varepsilon_2' = 10.0$, $k' = 0.09$ as $S_2 = \varepsilon_2'\dot{x}_2(t) + x_2(t) - k'(x_3(t) - 2x_2(t) + x_1(t))$.

$N(\tau)$ curve constructed for τ varied at a step of 0.01 exhibits a clearly distinguished minimum at the value of τ equal exactly to the delay time (Fig. 1b). It should be noted that the time series was smoothed in order to reduce the number of noise-related extrema (in evaluation of the derivative) using a procedure of local approximation over seven neighboring points. The length $L(\varepsilon, k)$ of a broken line connecting sequential points (ordered with respect to the abscissa) on the plane of coordinates $(x_3(t - \tau_3), \varepsilon_3\dot{x}_3(t) + x_3(t) - k_2(x_2(t) - x_3(t)))$ was found to be minimum for $\varepsilon_3 = 0.98$ and $k_2 = 0.95$, which were tried at a 0.01 step (Fig. 1c). Figure 1d presents the multimodal function recovered for these values of parameters, which well coincides with the true nonlinear function in the Ikeda equation. Approximation of the recovered nonlinear function by a 12th order polynomial yielded the following approximate values of the remaining parameters: $\mu_3' = 19.31$ and $x_{03}' = 1.025$ ($\pi/3 \approx 1.047$).

Now let us consider a chain consisting of diffusion-coupled systems with delayed feedback described by the following equations:

$$\begin{aligned}\varepsilon_i\dot{x}_i(t) &= -x_i(t) + f_i(x_i(t-\tau_i)) \\ &+ k[x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)],\end{aligned} \qquad (5)$$

where i is the number of the chain element and k is the coupling coefficient. Synchronization of such systems was studied by Burić et al. [15, 16], but the problem of recovery of the equations of chains of coupled time-delay systems has not been described until now. In order to reconstruct equations of the elements of chain (5), we propose the following approach. The values of τ_i can be determined using the method described above, based on the statistical analysis of temporal intervals between extrema of the time series. It was established that this method of evaluation of the delay time reliably works in a very broad range of parameters of the interacting systems and the coupling coefficients. In order to reconstruct the nonlinear functions f_i and the parameters ε_i and k, we will use the time series of oscillations in three sequential elements: ith and the neighboring ones, $(i-1)$th and $(i+1)$th. As can be seen from Eq. (5), a manifold of points with the coordinates $(x_i(t - \tau_i), \varepsilon_i\dot{x}_i(t) + x_i(t) - k[k_{i+1}(t) - 2x_i(t) + x_{i-1}(t)])$ plotted on the plane will reproduce the function f_i. Since the quantities ε_i and k are not known a priori, we have to try various ε and k in certain intervals so as to provide a single-valued relationship on the indicated plane, which is possible only with a correct choice of the parameters. As a quantitative criterion of such a unique relationship in the search for ε_1 and k_2, we again use the minimum length $L(\varepsilon_i, k)$ of a broken line connecting sequential points (ordered with respect to the abscissa) on this plane. A minimum of the length of this line will correspond to the true values of ε_i and k, while the corresponding plot will reproduce the correct nonlinear function.

In order to illustrate the proposed method, let us consider the reconstruction of an element of chain (5) consisting of three diffusion-coupled nonidentical Mackay–Glass systems

$$\dot{x}_i(t) = -b_i x_i(t) + \frac{a_i x_i(t-\tau_i)}{1 + x_i^{c_i}(t-\tau_i)} \qquad (6)$$

($i = 1, 2, 3$) with periodic boundary conditions ($x_4 \equiv x_1$). The system parameters ($\tau_1 = 300$, $\tau_2 = 350$, $\tau_3 = 400$; $a_1 = 0.2$, $a_2 = 0.3$, $a_3 = 0.25$; $b_{1,2,3} = 0.1$; $c_{1,2,3} = 10$) correspond to the motion of elements on a chaotic attractor of high dimensionality. Being divided by b_i, the chain of Mackay–Glass equations reduces to a system of type (5) with $\varepsilon_{1,2,3} = 10$. Note that the coefficient of coupling between elements of the chain ($k = 0.1$) is 2.5 times smaller than the value corresponding to the complete synchronization of oscillations in the chain.

As above, all the three coupled Mackay–Glass systems were rendered noisy by adding 10% white Gaussian noise (which corresponds to signal-to-noise ratio about 20 dB).

Figure 2 shows the results of reconstruction of the second element in the closed chain of three diffusion-coupled Mackay–Glass equations. Despite the presence of noise, the $N(\tau)$ curve constructed using the time series of $x_2(t)$ for τ varied at a step of 1.0 exhibits a clearly distinguished minimum at the value of $\tau = \tau_2 = 350$ (Fig. 2a). The length $L(\varepsilon, k)$ of a broken line exhibited a minimum for $\varepsilon_2 = 10.0$ and $k = 0.09$. Figure 2b presents the reconstructed nonlinear function f_2. Analogous procedures ensure a high-quality recovery of the first and third elements of the chain.

The proposed approach has no limitations with respect to the number of elements in the chain of coupled time-delay systems. Moreover, the method can be expanded to coupled delay-feedback systems of high order with several delay times, and even with additional delay in the coupling between local elements. However, such complications of the system lead to a considerable of the volume of computations, since a greater number of parameters have to be recovered. In the case of synchronization in the chain of diffusion-coupled systems with delayed feedback, which takes place for a strong coupling between elements, the proposed method ensures the recovery of parameters of the local elements, but the coupling coefficients cannot be determined.

In conclusion, we proposed a method for the recovery of model delay-differential equations for the chains of coupled delay-feedback systems, which is effective at a high level of noise.

Acknowledgments. This study was supported in part by the Russian Foundation for Basic Research, project no. 07-02-00589.

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Translated by P. Pozdeev