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Recovery of Equations of Coupled Time-Delay Systems from Time Series

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Abstract—A method is described for reconstructing the model differential equations with a delayed argument from time series of coupled systems with time-delay feedback. Using this method, it is also possible to establish the presence of coupling between time-delay systems and to determine the type, direction, and magnitude of coupling. The performance of the proposed method is illustrated by application to the experimental time series of coupled oscillators with time-delay feedback. *© 2005 Pleiades Publishing, Inc.*

Introduction. In recent years, the problem of reconstruction of the equations of nonlinear dynamical systems with time-delay feedback (time-delay systems) from their time series has received much attention [1–7]. The importance of this problem is related to the fact that time-delay systems are frequently encountered in nature [8]. The behavior of such systems is not entirely determined by the present state, but depends on the preceding states as well. Accordingly, the time-delay systems are usually described in terms of differential equations with a delayed argument. Such models are successfully used in various fields of physics, biology, physiology, and chemistry.

In some practically important cases, we are dealing with time-delay systems interacting with one another. In particular, such coupled systems are used in description of the dynamics of interacting populations [9–11] and in simulations of processes occurring in the human cardiovascular system [12, 13]. There are good prospects for using such models in the analysis of communication systems for secure data transmission [14], including those based on optical-feedback lasers [15–17]. However, the problem of reconstruction of the model equations of coupled time-delay system from their time series remains practically unstudied, although this situation is encountered in solving many important practical problems.

In this Letter, we describe a method capable of reconstructing the equations of coupled systems from their chaotic time series, determining the character of coupling (not known a priori), and finding the coupling coefficients.

Description of coupled systems. Consider two time-delay systems, X_1 and X_2, described in the absence of coupling by first-order differential equations with delayed argument of the following general type:

$$\varepsilon_{1,2}\dot{x}_{1,2}(t) = -x_{1,2}(t) + f_{1,2}(x_{1,2}(t-\tau_{1,2})), \quad (1)$$

where $x_{1,2}$ are the dynamical variables at the time t, $\tau_{1,2}$ are the delay times, $f_{1,2}$ are nonlinear functions, $\varepsilon_{1,2}$ are the parameters characterizing the inertial properties of the systems (subscripts 1 and 2 assign the variables and parameters to the systems X_1 and X_2, respectively). In the general case, Eq. (1) represents the mathematical model of an oscillatory system known in radio engineering as a delay-feedback oscillator [18], comprising a circuit with three ideal elements: nonlinear device, inertial element, and delay line. Below, these elements will be denoted by f, ε, and τ, respectively.

The time-delay systems X_1 and X_2 can be coupled in different ways. We will distinguish three types of coupling, by which the variable $x_1(t)$ of system X_1 is introduced with a certain coefficient k_1 into various points of circuit X_2 denoted by Arabic numerals *1–3* in Fig. 1. By the same token, the variable $x_2(t)$ of system X_2 can be introduced with the coefficient k_2 into various points of circuit X_1 denoted by Roman numerals I–III in Fig. 1. In the case when the types of coupling X_1 to X_2 and X_2 to X_1 are the same, the dynamics of these coupled systems is described by one of the following equations:

$$\varepsilon_{1,2}\dot{x}_{1,2}(t) = -x_{1,2}(t) + f_{1,2}(x_{1,2}(t-\tau_{1,2}) + k_{2,1}x_{2,1}(t-\tau_{1,2})), \quad (2)$$

$$\varepsilon_{1,2}\dot{x}_{1,2}(t) = -x_{1,2}(t) + f_{1,2}(x_{1,2}(t-\tau_{1,2}) + k_{2,1}x_{2,1}(t)), \quad (3)$$

$$\varepsilon_{1,2}\dot{x}_{1,2}(t) = -x_{1,2}(t) + f_{1,2}(x_{1,2}(t-\tau_{1,2})) + k_{2,1}x_{2,1}(t), \quad (4)$$

where $k_{1,2}$ are the coupling coefficients. Here, Eq. (2) describes the type of coupling referred to below as *1*/I, whereby the first time-delay system acts upon the sec-

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ond system at point *1* and the second time-delay system acts upon the first system at point I. By the same token, Eqs. (3) and (4) describe the mutually interacting systems coupled to each other at points *2*/II and *3*/III, respectively. If the types of mutual coupling of systems X_1 and X_2 are different, the common system will be described by the equations of different types. For example, the type of coupling denoted *1*/II implies that system X_1 is described by Eq. (3), while system X_2 is described by Eq. (2).

Description of the method. First, let us reconstruct the equation of system X_1, that is, determine the parameters τ_1, ε_1, and k_2 and find the function f_1 from the available time series. In order to determine the delay time τ_1 from the observed time series $x_1(t)$, we use the method developed in [5], where it was demonstrated that time series of the systems of type (1) contain virtually no extrema spaced from each other by τ_1. In order to find τ_1, we have to indicate extrema in the initial time series, determine the numbers N of the pairs of extrema in this series spaced by various times τ, and construct the $N(\tau)$ function. Then, the delay time τ_1 is determined by the position of the absolute minimum of the $N(\tau)$ function. The results of our investigations showed that this approach can be also successfully used in cases when system X_1 occurs under the action of another system X_2, provided that this external action does not lead to the appearance of a large number of additional extrema in the time realizations of oscillations in system X_1.

For determining the parameter ε_1 and the function f_1 of system X_1, as well as the coupling coefficient k_2, we propose a method based on an analysis of the time series of both observables $x_1(t)$ and $x_2(t)$. First, let us assume that the type of action of system X_2 upon system X_1 (i.e., the structure of equation describing dynamics of the time-delay system X_1 under the external action) is known. For example, consider the coupling type described by Eq. (2), whereby the variable of system X_2 is introduced into the feedback circuit of X_1 before the delay element (i.e., at point I in Fig. 1). In this case, Eq. (2) for system X_1 can be written as

$$\varepsilon_1\dot{x}_1(t) + x_1(t) = f_1(x_1(t-\tau_1) + k_2x_2(t-\tau_1)). \quad (5)$$

As can be seen from Eq. (5), a set of points with the coordinates $(x_1(t-\tau_1) + k_2x_2(t-\tau_1), \varepsilon_1\dot{x}_1(t) + x_1(t))$ plotted on the plane will reproduce the function f_1. Since the quantities ε_1 and k_2 are not known a priori, we have to plot $\varepsilon\dot{x}_1(t) + x_1(t)$ versus $x_1(t-\tau_1) + kx_2(t-\tau_1)$ for various ε and k in search for the single-valued relationship that is possible only for $\varepsilon = \varepsilon_1$ and $k = k_2$. As a quantitative criterion of such a unique relationship in the search for ε_1 and k_2, we can use the minimum length of a segment $L(\varepsilon, k)$ connecting points (ordered with respect to the abscissa) on the above plane. A minimum

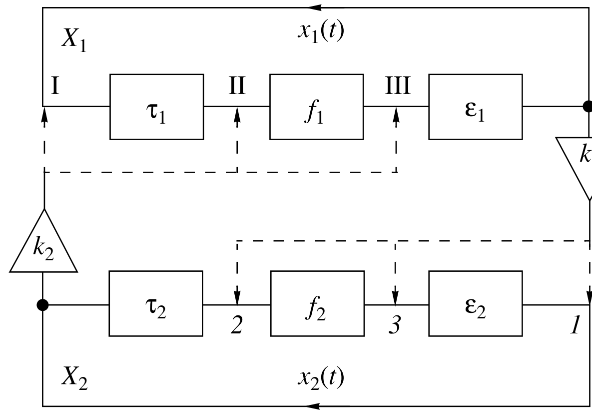

Fig. 1. Schematic diagram of coupled time-delay systems X_1 and X_2. Elements denoted by $\tau_{1,2}$, $f_{1,2}$, and $\varepsilon_{1,2}$ represent the delay line, nonlinear device, and inertial transformation of oscillations, respectively; elements $k_{1,2}$ determine the mutual coupling of systems X_1 and X_2. Points *1–3* (I–III) represent various types of introduction of the action of system X_1 into system X_2 (X_2 into system X_1).

of $L(\varepsilon, k)$ ($L_{\min}(\varepsilon, k)$) will correspond to $\varepsilon = \varepsilon_1$ and $k = k_2$, while the dependence of $\varepsilon_1\dot{x}_1(t) + x_1(t)$ on $x_1(t-\tau_1) + k_2x_2(t-\tau_1)$ constructed for these parameters will reproduce a certain nonlinear function that can be (if necessary) approximated. The proposed approach employs all points of the time series, which allows short time series to be used for reconstruction of the system parameters ε_1 and k_2 and the nonlinear function f_1.

The same method can be used for reconstructing the nonlinear function f_1 and the parameters ε_1 and k_2 in the situations described by Eqs. (3) and (4), by plotting $\varepsilon\dot{x}_1(t) + x_1(t)$ versus $x_1(t-\tau_1) + kx_2(t)$ and $\varepsilon\dot{x}_1(t) + x_1(t) - kx_2(t)$ versus $x_1(t-\tau_1)$, respectively, for various ε and k. If the point (I, II, or III) at which system X_2 acts upon system X_1 is not known a priori, it is necessary to perform reconstruction for each of the three model equations (2)–(4) for system X_1 and determine the corresponding values of $L_{\min}(\varepsilon, k)$. The only correct structure of the model equation will be indicated by single-valued form of the reconstructed function and, accordingly, by the lowest of the three values of $L_{\min}(\varepsilon, k)$. Thus, the proposed method allows both the parameters of mutually coupled time-delay systems and the type of this action (i.e., the form of the model equation) to be reconstructed from the observed time series.

The same procedures are used to reconstruct the time-delay system X_2 from the available time series of variables $x_2(t)$ and $x_1(t)$. These procedures yield the parameters τ_2 and ε_2, define the form of the nonlinear function f_2 for system X_2, and determine the coupling coefficient k_1 and the type of action of system X_1 upon system X_2.

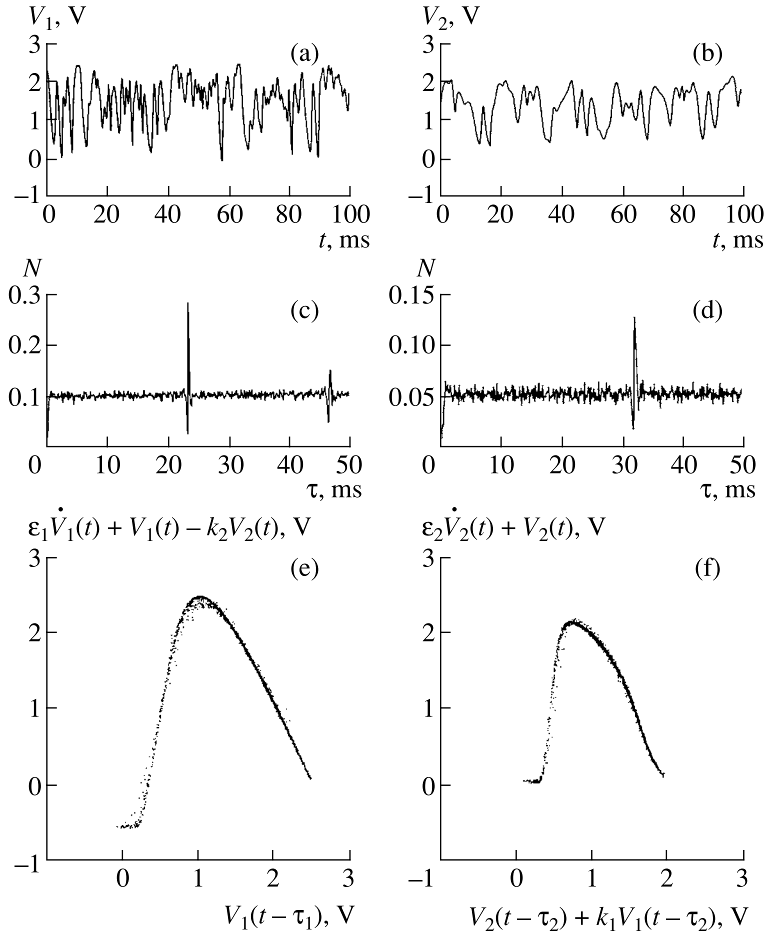

Fig. 2. Reconstruction of the system of two delay-feedback oscillators coupled in the *1*/III type: (a, b) typical experimental time series of systems X_1 and X_2, respectively; (c, d) plots of the number N of the pairs of extrema in a time series of X_1 and X_2, respectively, spaced by various times τ, normalized to the total number of such extrema; (e) reconstruction of the nonlinear function f_1 (for τ_1 = 23.0 ms, ε_1 = 0.46 ms, and k_2 = 0.10); (f) reconstruction of the nonlinear function f_2 (for τ_2 = 31.7 ms, ε_2 = 1.06 ms, and k_1 = –0.10).

Verification of the method. We have verified the proposed reconstruction method by applying the procedure outlined above to the experimental time series of two coupled delay-feedback oscillators. When such coupled oscillators are schematically depicted as in Fig. 1, the delay of signal $x_1(t)$ by the time τ_1 and the delay of signal $x_2(t)$ by the time τ_2 are provided by the corresponding delay line; the role of a nonlinear element in each oscillator is played by an amplifier with a transmission function f_1 or f_2; the inertial properties are determined by a filter, the parameters of which define ε_1 or ε_2; and the coupling is effected by adding amplifiers ensuring the transmission coefficients k_1 or k_2. In the example under consideration, the coupling type according to our classification corresponded to *1*/III. In the case when the inertial element is a first-order low-frequency *RC* filter, such oscillators in the absence of coupling are described by the equation

$$R_{1,2}C_{1,2}\dot{V}_{1,2}(t) = -V_{1,2}(t) + f_{1,2}(V_{1,2}(t-\tau_{1,2})), \quad (6)$$

where $V_{1,2}(t)$ and $V_{1,2}(t-\tau_{1,2})$ are the voltages at the input and output of the corresponding delay line; and $R_{1,2}$ and $C_{1,2}$ are the resistances and capacitances in the filters of the corresponding oscillator, respectively. Equation (6) has the form of Eq. (1) with $\varepsilon_{1,2} = R_{1,2}C_{1,2}$.

The signals $V_1(t)$ and $V_2(t)$ at the input of the corresponding delay line were recorded and digitized using an analog-to-digital converter at sampling rate of 10 kHz for the following parameters of coupled oscillators: τ_1 = 23 ms; τ_2 = 31.7 ms; R_1C_1 = 0.48 ms; R_2C_2 = 1.01 ms; k_1 = –0.1; k_2 = 0.1. Fragments of the corresponding time series are presented in Figs. 2a and 2b. Upon counting the number of situations where the signals $\dot{V}_1(t)$ and $\dot{V}_1(t-\tau)$ simultaneously vanish for various τ (tried at a step equal to the period of sampling T_s = 0.1 ms), we construct the function $N(\tau)$ (Fig. 2c). In order to evaluate the derivative $\dot{V}_1(t)$ from the time series, we used a local parabolic approximation. The absolute minimum of $N(\tau)$ is observed for the first oscillator at τ = 23.0 ms (Fig. 2c) and for the second oscillator at τ = 31.7 ms (Fig. 2d).

During the reconstruction of oscillator X_1 in the form of Eq. (4), the curve of $L(\varepsilon, k)$, constructed using a 0.01 ms step in ε and a 0.01 step in k, exhibited a minimum at ε = 0.46 ms and k = 0.10 (which provides a quite close estimation of the ε_1 and k_2 values). The reconstructed nonlinear function (Fig. 2e) shows a good coincidence with the true transmission characteristic f_1 of the nonlinear element of the first oscillator.

A minimum of the curve of $L(\varepsilon, k)$ constructed for system X_2 represented by Eq, (2) was observed at ε = 1.06 ms and k = –0.10 (also sufficiently close to the true values of ε_2 and k_1). The reconstructed nonlinear function f_2 is depicted in Fig. 2f.

Conclusion. We proposed a method for reconstructing coupled time delay systems using the observable time series. This method allows the delay times, inertial parameters, nonlinear functions, and coupling coefficients for the two coupled time-delay systems to be determined, even when the type of coupling is a priori unknown. In the latter case, the reconstruction procedure makes it possible to establish the type of coupling between the time-delay systems. In contrast to other methods of determining the coupling between systems using their time series [19–21], the proposed procedure is capable of determining not only the direction of coupling, but its magnitude as well.

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REFERENCES

1. M. J. Bünner, M. Popp, Th. Meyer, *et al.*, Phys. Rev. E **54**, 3082 (1996).
2. H. Voss and J. Kurths, Phys. Lett. A **234**, 336 (1997).
3. R. Hegger, M. J. Bünner, H. Kantz, and A. Giaquinta, Phys. Rev. Lett. **81**, 558 (1998).
4. M. J. Bünner, M. Ciofini, A. Giaquinta, *et al.*, Eur. Phys. J. D **10**, 165 (2000).
5. B. P. Bezruchko, A. S. Karavaev, V. I. Ponomarenko, and M. D. Prokhorov, Phys. Rev. E **64**, 056216 (2001).
6. V. I. Ponomarenko and M. D. Prokhorov, Pis'ma Zh. Tekh. Fiz. **28** (16), 37 (2002) [Tech. Phys. Lett. **28**, 680 (2002)].
7. V. S. Udaltsov, J.-P. Goedgebuer, L. Larger, *et al.*, Phys. Lett. A **308**, 54 (2003).
8. J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations* (Springer, New York, 1993).
9. Y. Kuang, *Delay Differential Equations With Applications in Population Dynamics* (Academic Press, Boston, 1993).
10. G. A. Bocharov and F. A. Rihan, J. Comp. Appl. Math. **125**, 183 (2000).
11. Y. Song, M. Han, and Y. Peng, Chaos, Solitons and Fractals **22**, 1139 (2004).
12. H. Seidel and H. Herzel, Physica D **115**, 145 (1998).
13. K. Kotani, K. Takamasu, Y. Ashkenazy, *et al.*, Phys. Rev. E **65**, 051923 (2002).
14. B. Mensour and A. Longtin, Phys. Lett. A **244** (1–3), 59 (1998).
15. V. S. Udaltsov, J.-P. Goedgebuer, L. Larger, and W. T. Rhodes, Phys. Rev. Lett. **86**, 1892 (2001).
16. I. V. Koryukin and P. Mandel, Phys. Rev. E **65**, 026201 (2002).
17. E. M. Shahverdiev, S. Sivaprakasam, and K. A. Shore, Phys. Rev. E **66**, 017206 (2002).
18. A. S. Dmitriev and V. Ya. Kislov, *Stochastic Oscillations in Radio Physics and Electronics* (Nauka, Moscow, 1989) [in Russian].
19. M. G. Rosenblum and A. S. Pikovsky, Phys. Rev. E **64**, 045202 (2001).
20. M. G. Rosenblum, L. Cimponeriu, A. Bezerianos, *et al.*, Phys. Rev. E **65**, 041909 (2002).
21. D. A. Smirnov and B. P. Bezruchko, Phys. Rev. E **68**, 046209 (2003).

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