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Estimation of coupling between time-delay systems from time series

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We propose a method for estimation of coupling between the systems governed by scalar time-delay differential equations of the Mackey-Glass type from the observed time series data. The method allows one to detect the presence of certain types of linear coupling between two time-delay systems, to define the type, strength, and direction of coupling, and to recover the model equations of coupled time-delay systems from chaotic time series corrupted by noise. We verify our method using both numerical and experimental data.

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I. INTRODUCTION

The problem of recovery of nonlinear dynamical models of time-delay systems from time series has received much attention in recent years [1–6]. However, the reconstruction of model equations of coupled time-delay systems and estimation of coupling between them from time series have not been practically considered yet. At the same time, interaction between time-delay systems is a typical case in many important problems. For example, the use of coupled time-delay systems demonstrating chaotic dynamics of a very high dimension is promising for secure communication [7] including chaotic communication systems based on lasers with optical feedback [8–10]. Besides, coupled time-delay differential equations are used for the description of behavior of interacting populations [11–13] and for modeling the processes in the human cardiovascular system [14,15]. In this paper we propose a method that is able to reconstruct two coupled scalar time-delay systems and to estimate the coupling strength and direction from the observed time series data.

The proposed method is suitable for time-delay systems X_1 and X_2 described in the absence of coupling by the first-order delay-differential equation with single delay time

$$\varepsilon_{1,2}\dot{x}_{1,2}(t) = -x_{1,2}(t) + f_{1,2}(x_{1,2}(t-\tau_{1,2})), \tag{1}$$

where $\tau_{1,2}$ are the delay times, functions $f_{1,2}$ define nonlocal correlations in time, and the parameters $\varepsilon_{1,2}$ characterize the inertial properties of the first and the second system, respectively. In general case Eq. (1) is a mathematical model of an oscillating system composed of a ring with three ideal elements: nonlinear, inertial, and delay. In Fig. 1 these elements are denoted as f_1, ε_1, and τ_1, respectively, for the ring system X_1 and as f_2, ε_2, and τ_2 for the ring system X_2.

The time-delayed feedback systems X_1 and X_2 can be coupled in different ways. For instance, the system X_1 variable $x_1(t)$ multiplied by a coupling coefficient k_1 can be injected into the ring system X_2 at one of the three points indicated in Fig. 1 by the Arabic numerals 1-3. Similarly, the system X_2 variable $x_2(t)$ multiplied by a coupling coefficient k_2 can be injected into the ring system X_1 at different points indicated in Fig. 1 by the Roman numerals I–III. If the type of action of X_1 on X_2 is the same as the type of action of X_2 on X_1, then the dynamics of both coupled systems is described by one of the following equations:

$$\varepsilon_{1,2}\dot{x}_{1,2}(t) = -x_{1,2}(t) + f_{1,2}(x_{1,2}(t-\tau_{1,2}) + k_{2,1}x_{2,1}(t-\tau_{1,2})), \tag{2}$$

$$\varepsilon_{1,2}\dot{x}_{1,2}(t) = -x_{1,2}(t) + f_{1,2}(x_{1,2}(t-\tau_{1,2}) + k_{2,1}x_{2,1}(t)), \tag{3}$$

$$\varepsilon_{1,2}\dot{x}_{1,2}(t) = -x_{1,2}(t) + f_{1,2}(x_{1,2}(t-\tau_{1,2})) + k_{2,1}x_{2,1}(t). \tag{4}$$

Equation (2) governs the both systems X_1 and X_2 for the type of coupling at which the first time-delay system acts on the second one at the point 1 and the second system acts upon the first one at the point I. We denote this type of coupling as 1/I. Equations (3) and (4) describe both coupled systems for the types of coupling 2/II and 3/III, respectively. A block diagram of the coupled time-delay systems for the coupling type 3/III is shown in Fig. 2. If the systems X_1 and X_2 affect on each other in different ways, then they are described by different equations. For example, in the case of 1/II type of coupling, the system X_1 is given by Eq. (3) and the system X_2 is given by Eq. (2).

Certainly, the variety of possible types of coupling between time-delay systems is very large. In this paper we

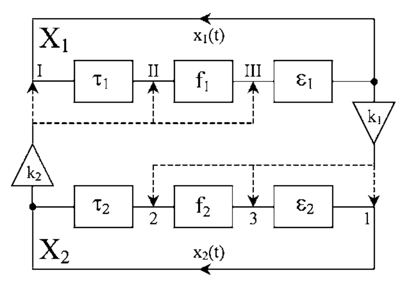

FIG. 1. Block diagram of two coupled time-delay systems X_1 and X_2. The elements denoted as τ_1 and τ_2 provide a delay and the elements denoted as f_1 and f_2, and ε_1 and ε_2 provide the nonlinear and inertial transformations of oscillations, respectively. The elements k_1 and k_2 determine the strength of coupling between X_1 and X_2. Arabic numerals 1–3 designate points where the system X_1 acts on the system X_2. Roman numerals I–III designate points where X_2 acts on X_1.

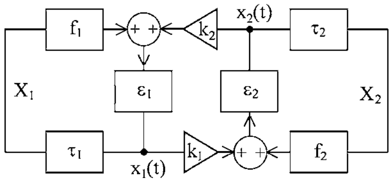

FIG. 2. Block diagram of coupled time-delay systems for the 3/III type of coupling.

restrict our consideration to only three chosen types of linear coupling between two time-delay systems and propose for them the method of coupling coefficients estimation from time series. The possibility of method extension to a more wide class of coupled time-delay systems and the method restrictions are discussed in the conclusion of the paper.

The paper is organized as follows. In Sec. II the method for estimation of the parameters of coupled time-delay systems having the form of Eqs. (2)–(4) is presented. In Sec. III we apply the method to various simulated and experimental time series. The obtained results are summarized and discussed in Sec. IV.

II. METHOD DESCRIPTION

The proposed method for estimation of coupling between time-delay systems is based on the reconstruction of the time-delay systems X_1 and X_2 from time series. At first we recover the model equation of the system X_1, i.e., we estimate the parameters τ_1, ε_1, and k_2 and reconstruct the nonlinear function f_1.

To recover the delay time τ_1 from the temporal realization $x_1(t)$ we exploit the method proposed in Ref. [5], where we have shown that there are practically no extrema separated in time by τ_1 in the time series of time-delay system (1). Actually, differentiation of Eq. (1) with respect to t gives the following equation for the system X_1:

$$\varepsilon_1 \ddot{x}_1(t) = -\dot{x}_1(t) + \frac{df_1(x_1(t-\tau_1))}{dx_1(t-\tau_1)} \dot{x}_1(t-\tau_1). \qquad (5)$$

If for $\dot{x}_1(t)=0$ in a typical case of quadratic extrema $\ddot{x}_1(t) \neq 0$, then for $\varepsilon_1 \neq 0$ the condition $\dot{x}_1(t-\tau_1) \neq 0$ must be fulfilled. Thus, there must be no extremum separated in time by τ_1 from a quadratic extremum [5]. Then, for τ_1 definition one has to determine the extrema in the time series and after that to define for different values of time τ the number N of pairs of extrema separated in time by τ and to construct the $N(\tau)$ plot. This plot will demonstrate a pronounced minimum at time τ_1 equal to the delay time of the system. We find that this method of the delay time estimation can be successfully applied in the case where the system X_1 is affected by the system X_2 under the condition that this action is not followed by the appearance of a great number of additional extrema in the time series of X_1. Note that this technique uses only operations of comparing and adding for the delay time definition. It needs neither calculation of approximation error [3,6], nor calculation of any measure of complexity of the trajectory [1,4] and therefore it does not need significant time of computation.

To recover the parameter ε_1, the nonlinear function f_1, and the coupling coefficient k_2 we propose a method using time series of both variables $x_1(t)$ and $x_2(t)$. At first, let us assume that the type of action of X_2 on X_1 is known *a priori*, i.e., we know the form of equation governing the dynamics of the time-delay system X_1. As an example, we consider the case described by Eq. (2), when the system X_2 variable is injected into the time-delayed feedback system X_1 before the element providing the delay (point I in Fig. 1). Let us write Eq. (2) for the system X_1 as

$$\varepsilon_1 \dot{x}_1(t) + x_1(t) = f_1\big(x_1(t-\tau_1) + k_2 x_2(t-\tau_1)\big). \qquad (6)$$

According to Eq. (6) it is possible to recover the function f_1 by plotting in a plane a set of points with coordinates $[x_1(t-\tau_1)+k_2x_2(t-\tau_1), \varepsilon_1\dot{x}_1(t)+x_1(t)]$. Since the parameters ε_1 and k_2 are unknown, one needs to plot $\varepsilon\dot{x}_1(t)+x_1(t)$ versus $x_1(t-\tau_1)+kx_2(t-\tau_1)$ under variation of ε and k, searching for a single-valued dependence in the plane $[x_1(t-\tau_1)+kx_2(t-\tau_1), \varepsilon\dot{x}_1(t)+x_1(t)]$, which is possible only for $\varepsilon=\varepsilon_1$ and $k=k_2$. As a quantitative criterion of single-valuedness in searching for ε_1 and k_2 we use the minimal length of a line $L(\varepsilon,k)$, connecting all points ordered with respect to the abscissa in the mentioned plane. The similar criteria of quality for time-delay system recovery, based on calculation of measure of complexity of the projected time series were used in Refs. [1,16] for the reconstruction of single time-delay systems. The minimum $L_{min}(\varepsilon,k)$ is observed at $\varepsilon=\varepsilon_1$ and $k=k_2$. The dependence of $\varepsilon_1\dot{x}_1(t)+x_1(t)$ on $x_1(t-\tau_1)+k_2x_2(t-\tau_1)$ for the defined ε_1 and k_2 reproduces the nonlinear function that can be approximated if necessary. The proposed technique uses all points of the time series. It allows one to estimate the parameters ε_1 and k_2 and to reconstruct the nonlinear function from short time series.

Similarly it is possible to recover the nonlinear function f_1 and the parameters ε_1 and k_2 for the system X_1 described by Eq. (3) or Eq. (4) by plotting $\varepsilon\dot{x}_1(t)+x_1(t)$ versus $x_1(t-\tau_1)+kx_2(t)$ or $\varepsilon\dot{x}_1(t)+x_1(t)-kx_2(t)$ versus $x_1(t-\tau_1)$, respectively, under variation of ε and k. If we know that time-delay systems (1) are linearly coupled in one of the three ways described in Sec. I, but we do not know at which point (I, II, or III) X_2 acts on X_1, we have to reconstruct each of the model equations (2)–(4) of the system X_1 and to define $L_{min}(\varepsilon,k)$ for each of these equations. The single valuedness of the recovered nonlinear function can be achieved only in the case of the true choice of the model equation. Hence the smallest $L_{min}(\varepsilon,k)$ from the three obtained ones will correspond to the true model choice. Thus, along with estimation of the parameters of coupled time-delay systems, the method allows one to identify the type of coupling.

The time-delay system X_2 can be reconstructed from the time series of $x_2(t)$ and $x_1(t)$ in a similar way. The method allows us to estimate the parameters τ_2 and ε_2, to recover the nonlinear function f_2, and to define the coupling coefficient k_1 and the type of action of X_1 on X_2. Identifying the type of coupling between the systems and estimating the values of both coupling coefficients k_1 and k_2 we can judge the character of interaction between the time-delay systems X_1 and X_2.

III. METHOD APPLICATION

A. Estimation of coupling between identical Mackey-Glass equations

First we apply the method to the time series produced by two coupled identical time-delay systems described in the absence of coupling by the Mackey-Glass equation

$$\dot{x}_{1,2}(t) = -b_{1,2}x_{1,2}(t) + \frac{a_{1,2}x_{1,2}(t-\tau_{1,2})}{1+x_{1,2}^{c_{1,2}}(t-\tau_{1,2})}, \qquad (7)$$

which can be converted to Eq. (1) with $\varepsilon_{1,2}=1/b_{1,2}$ and the function

$$f_{1,2}(x_{1,2}(t-\tau_{1,2})) = \frac{a_{1,2}x_{1,2}(t-\tau_{1,2})}{b_{1,2}(1+x_{1,2}^{c_{1,2}}(t-\tau_{1,2}))}. \qquad (8)$$

The types of action of the systems X_1 and X_2 on each other are chosen to be the same (Fig. 2). We use the 3/III type of coupling according to our classification. In this case the dynamics of both coupled systems is governed by Eq. (4). The system parameters are chosen to be $\tau_{1,2}=300, a_{1,2}=0.2, b_{1,2}=0.1, c_{1,2}=10, k_1=0.05$, and $k_2=0.1$ to produce a dynamics on a high-dimensional chaotic attractor. Part of the time series of the system X_1 is shown in Fig. 3(a). The time series is sampled in such a way that 300 points in time series cover a period of time equal to the delay time $\tau_1=300$. The data set consists of 10000 points and exhibits about 600 extrema as well as the time series of the system X_2.

For various τ values we count the number N of situations when $\dot{x}_1(t)$ and $\dot{x}_1(t-\tau)$ are simultaneously equal to zero, normalize N to the total number of extrema in the time series, and construct the $N(\tau)$ plot [Fig. 3(b)]. The step of τ variation in Fig. 3(b) is equal to unity. The time derivative $\dot{x}_1(t)$ and the extremal points are defined from the time series by applying a local parabolic approximation. In the case of noise absence we use three points in the procedure of the local fit. The pronounced minimum of $N(\tau)$ takes place exactly at $\tau=\tau_1=300$.

The $L(\varepsilon,k)$ plot [Fig. 3(c)] allows us to recover the parameters ε_1 and k_2. To reduce the computation time we choose a large initial step of ε and k variation and then reduce it in the neighborhood of minimum $L(\varepsilon,k)$. In Fig. 3(c) the step of ε variation is set by 0.1 and the step of k variation is set by 0.01. The minimum of $L(\varepsilon,k)$ is observed at $\varepsilon=10.1$ and $k=0.10$. These values agree well with the true parameter values $\varepsilon_1=1/b_1=10$ and $k_2=0.1$. In Fig. 3(d) the recovered nonlinear function f_1 is shown. It coincides practically with the true function (8). Note that for the construction of the $L(\varepsilon,k)$ plot and for the recovery of the function f_1 we use only 2000 points of the time series of $x_1(t)$ and $x_2(t)$.

In a similar way we reconstruct the time-delay system X_2 and obtain the following estimation of its parameters: $\tau_2=300, \varepsilon_2=10.1$, and $k_1=0.05$. For the indicated above parameter values of two coupled identical systems (7) the method provides the detection of coupling presence and high accuracy of coupling coefficients estimation at $0.003\leq k_{1,2}\leq 0.3$. It should be noted that the method is still efficient for sufficiently high levels of noise. For example, we apply the method to the data produced by adding a zero-mean Gaussian white noise to the time series of both coupled identical Mackey-Glass equations. For the case where the additive noise has a standard deviation of up to 20% of the standard deviation of the data without noise, we obtain the same values of the recovered parameters as in the considered above case of noise absence. Note that searching for the extremal points in the case of noise level of 20% we use seven points for local parabolic approximation of the data. However, the quality of the nonlinear function recovery deteriorates with the noise increasing. If the noise is involved in the system dynamics the exact definition of the parameters becomes impossible for smaller noise levels than in the case of additive noise.

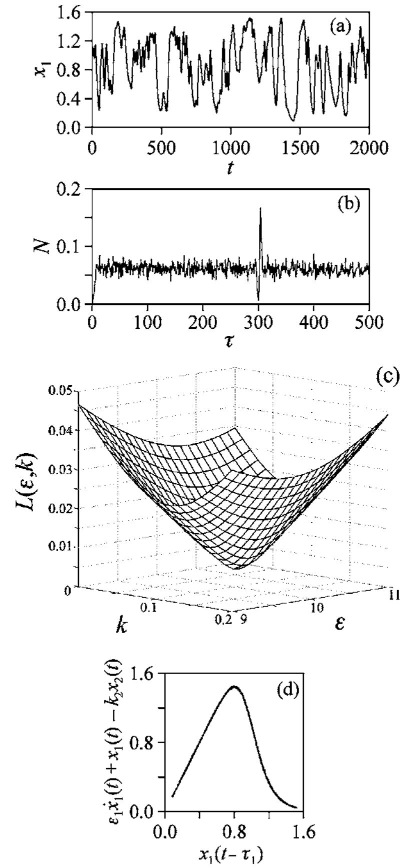

FIG. 3. Reconstruction of the Mackey-Glass system X_1 coupled with the identical Mackey-Glass system X_2 for the 3/III type of coupling. (a) The time series of the system X_1. (b) Number N of pairs of extrema in the time series of X_1 separated in time by τ, as a function of τ. $N(\tau)$ is normalized to the total number of extrema in the time series. (c) The $L(\varepsilon,k)$ plot for the choice of the model equation in the form of Eq. (4). $L(\varepsilon,k)$ is normalized to the number of points. $L_{min}(\varepsilon,k)=L(10.1,0.10)$. (d) The recovered nonlinear function at $\tau_1=300, \varepsilon_1=10.1$, and $k_2=0.1$.

B. Estimation of coupling between nonidentical Mackey-Glass equations in the presence of noise

Let us consider a more general case of coupled nonidentical noisy time-delay systems X_1 and X_2 with different types of action on each other. We apply the method to the time

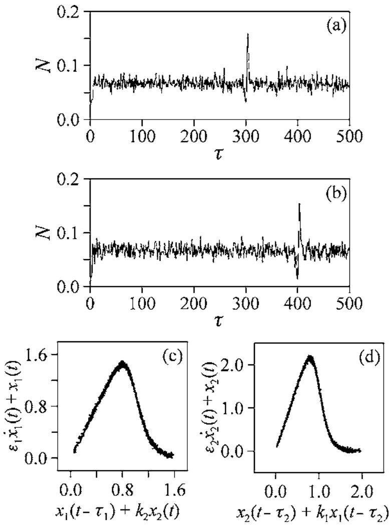

FIG. 4. Reconstruction of coupled nonidentical Mackey-Glass systems from data corrupted by additive Gaussian white noise for noise level of 10% and 1/II type of coupling. (a) Number N of pairs of extrema in the system X_1 time series separated in time by τ normalized to the total number of extrema. (b) Number N of pairs of extrema in the system X_2 time series separated in time by τ normalized to the total number of extrema. (c) The recovered nonlinear function f_1 at $\tau_1=300, \varepsilon_1=10.0$, and $k_2=0.10$. (d) The recovered nonlinear function f_2 at $\tau_2=400, \varepsilon_2=10.1$, and $k_1=0.05$.

series of two coupled Mackey-Glass equations for the 1/II type of coupling (see Sec. I) at $\tau_1=300$, $\tau_2=400$, $a_1=0.2$, $a_2=0.3$, $b_{1,2}=0.1$, $c_{1,2}=10$, $k_1=0.05$, and $k_2=0.1$. To investigate the robustness of the method to perturbations we analyze the time series of X_1 and X_2 both corrupted by additive Gaussian white noise. Figure 4 illustrates the obtained results for a noise level of 10%.

The presence of noise in time series brings into existence spurious extrema. These extrema are not caused by the intrinsic dynamics of a system and temporal distances between them are random. To smooth the time series corrupted by noise and to reduce the number of extrema caused by noise we use more nearest-neighbor points in the procedure of local approximation while estimating derivatives from data in comparison with the case of noise absence [5]. For the considered case of noise level of 10% we use five points for the local parabolic fit. In spite of the noise presence the pronounced minimum of the $N(\tau)$ plot constructed for the system X_1 time series is observed at $\tau=\tau_1=300$ [Fig. 4(a)] and the pronounced minimum of $N(\tau)$ for the time series of X_2 is observed at $\tau=\tau_2=400$ [Fig. 4(b)]. The $L(\varepsilon,k)$ plot, constructed for the system X_1 recovery in the form of Eq. (3), demonstrates the minimum at $\varepsilon=10.0$ and $k=0.10$ giving the accurate estimation of ε_1 and k_2. The location of the absolute minimum of the $L(\varepsilon,k)$ plot, constructed for the system X_2 recovery in the form of Eq. (2), allows us to obtain the following estimation of the parameters: $\varepsilon_2=10.1$ and $k_1=0.05$. The recovered nonlinear functions f_1 and f_2 are presented in Figs. 4(c) and 4(d), respectively.

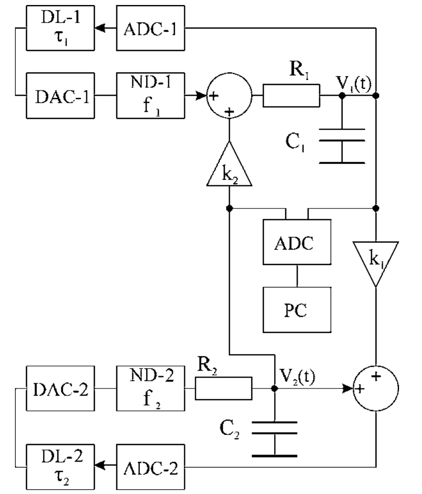

FIG. 5. Block diagram of the experimental system of coupled electronic oscillators with delayed feedback for the 1/III type of coupling. DL-1 and DL-2 are the delay lines, ND-1 and ND-2 are the nonlinear devices, ADC-1 and ADC-2 are the analog-to-digital converters, and DAC-1 and DAC-2 are the digital-to-analog converters of the first and the second oscillator, respectively. ADC is a two-channel analog-to-digital converter and PC is a computer.

C. Estimation of coupling between electronic oscillators with delayed feedback from experimental data

The next example is the method application to experimental time series gained from two coupled electronic oscillators with delayed feedback. A block diagram of the experimental setup is shown in Fig. 5. The delay of the signal $V_1(t)$ for time τ_1 and the delay of the signal $V_2(t)$ for time τ_2 are provided by the delay lines DL-1 and DL-2, respectively, constructed using digital elements or computer. The delay line DL-1 was constructed in the following way: the analog-to-digital converter ADC-1 and the digital-to-analog converter DAC-1 were constructed using the chips. The digital code was stored into RAM performed using the chip. In the RAM we constructed the FIFO buffer which length we were able to adjust. The entire circuitry of this device is too cumbersome to be included in the paper. It is available at http://www.nonlinmod.sgu.ru/doc/scheme.pdf. To follow the experiment one can construct the delay line using PC and ADC/DAC card. By this strategy the second delay line DL-2 was constructed. We used the ADC/DAC card L-205 and PC. The input voltage was read using a special program and stored into RAM. In the RAM the FIFO buffer was organized using a software. We were able to regulate the buffer length. The delay lines are practically dispersion free while the signal band defined by the filter parameters lies within the band of analog-to-digital converters. The conversion frequencies of analog-to-digital converters are about 100 kHz and the cutoff frequencies of the filters are about 1 kHz and 2 kHz.

The role of nonlinear devices, ND-1 and ND-2, is played in the oscillators by the amplifiers with the transfer functions f_1 and f_2, respectively. These nonlinear devices, ND-1 and ND-2, were constructed using bipolar transistors and field-effect transistors, respectively (Fig. 6). The inertial properties

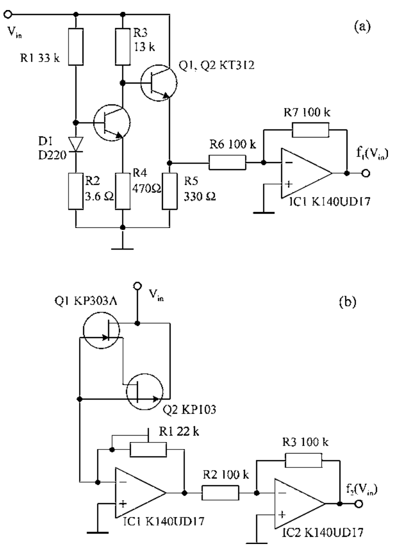

FIG. 6. Circuitries of the nonlinear devices ND-1 (a) and ND-2 (b).

of oscillators are defined by low-frequency first-order RC filters R_1C_1 and R_2C_2, which parameters specify ε_1 and ε_2. The coupling of oscillators is realized using summing amplifiers with gains k_1 and k_2. The type of coupling corresponds to the case 1/III according to our classification.

In the absence of coupling the considered oscillators are given by

$$R_{1,2}C_{1,2}\dot{V}_{1,2}(t) = -V_{1,2}(t) + f_{1,2}(V_{1,2}(t-\tau_{1,2})), \qquad (9)$$

where $V_{1,2}(t)$ and $V_{1,2}(t-\tau_{1,2})$ are the delay line input and output voltages, respectively, $R_{1,2}$ and $C_{1,2}$ are the resistances and capacitances of the filter elements in the first and the second oscillator, respectively. Equation (9) is of the form (1) with $\varepsilon_{1,2}=R_{1,2}C_{1,2}$.

We record the signals $V_1(t)$ and $V_2(t)$ using a two-channel analog-to-digital converter ADC (Fig. 5) with the sampling frequency $f_s=10$ kHz at $\tau_1=23$ ms, $\tau_2=31.7$ ms, $R_1C_1=0.48$ ms, $R_2C_2=1.01$ms, $k_1=-0.1$ and $k_2=0.1$. To measure the resistance and capacitance of the filter elements we used the universal digital multimeter E7–8 (Russia). It has the accuracy of 0.3% for the measurement of the capacitance and the accuracy of 0.15% for the measurement of the resistance. Taking into account the measurement errors, the values of R_1C_1 and R_2C_2 can be written as $R_1C_1=0.48$ ms± 0.003 ms and $R_2C_2=1.01$ ms± 0.005 ms.

The parts of the time series of the signals $V_1(t)$ and $V_2(t)$ are presented in Figs. 7(a) and 7(b), respectively. For the step of τ variation equal to the sampling time $T_s=0.1$ ms, the pronounced minimum of $N(\tau)$ takes place at $\tau=23.0$ ms [Fig. 7(c)] for the first oscillator and at $\tau=31.7$ ms [Fig. 7(d)] for the second oscillator.

To construct the $L(\varepsilon,k)$ plot we use the step of ε variation equal to 0.01 ms and the step of k variation equal to 0.01. Reconstructing the model of the oscillator X_1 in the form of Eq. (4) we obtain the minimum of $L(\varepsilon,k)$ at $\varepsilon=0.46$ ms and $k=0.10$ that are close to the true values of ε_1 and k_2. The recovered nonlinear function [Fig. 8(a)] coincides closely

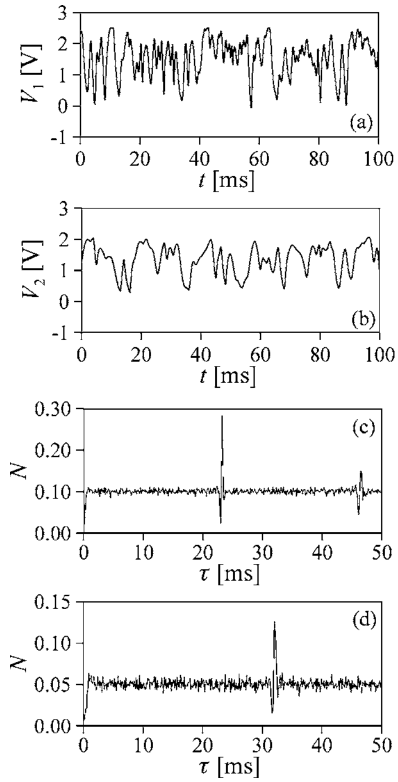

FIG. 7. Experimental time series of the first (a) and the second (b) coupled electronic oscillators with delayed feedback. Number N of pairs of extrema in the time series of the oscillator X_1 (c) and the oscillator X_2 (d), separated in time by τ, as a function of τ. $N(\tau)$ is normalized to the total number of extrema in the time series.

with the true transfer function f_1 of the nonlinear element of the first oscillator.

Reconstructing the system X_2 in the form of Eq. (2) we observe the minimum of $L(\varepsilon,k)$ at $\varepsilon=1.06$ ms and $k=-0.10$ that give a close estimation of ε_2 and k_1. In Fig. 8(b)

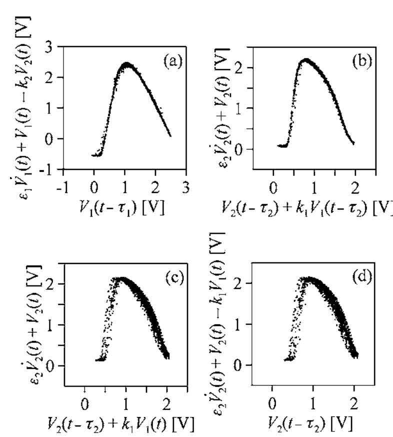

FIG. 8. Reconstruction of nonlinear functions of coupled electronic oscillators with delayed feedback. (a) The recovered nonlinear function f_1 at $\tau_1=23.0$ ms, $\varepsilon_1=0.46$ ms, and $k_2=0.10$. (b)–(d) Results of the nonlinear function f_2 recovery for the choice of the second oscillator model equation in the form of Eqs. (2)–(4), respectively, at the recovered parameters $\varepsilon_2=1.06$ ms and $k_1=-0.10$ (b), $\varepsilon_2=0.98$ ms and $k_1=0.00$ (c), and $\varepsilon_2=0.97$ ms and $k_1=0.01$ (d).

the recovered nonlinear function of the system X_2 is shown. This function coincides closely with the true transfer function f_2 of the nonlinear element of the second oscillator. If it is known that the system X_2 is governed by one of the three model equations (2)–(4) but it is not known exactly which of the three types of linear coupling takes place, then one has to recover each of the model equations (2)–(4) for the system X_2 and to define $L_{min}(\varepsilon,k)$ for each of the three cases. For the choice of the system X_2 model in the form of Eq. (3) we obtain $L_{min}(\varepsilon,k)=L(0.98\text{ ms},0.00)=0.135$. Reconstructing the model equation of the system X_2 in the form of Eq. (4) we obtain $L_{min}(\varepsilon,k)=L(0.97\text{ ms},0.01)=0.134$. In both cases $L_{min}(\varepsilon,k)$ is normalized to the number of points. The results of the nonlinear function f_2 recovery for the choice of the model equation in the form of Eqs. (3) and (4) are shown in Figs. 8(c) and 8(d), respectively. From the three plots presented in Figs. 8(b)–8(d) only the plot in Fig. 8(b) demonstrates a set of points that is close to a single-valued curve. In this case $L_{min}(\varepsilon,k)=L(1.06\text{ ms},-0.10)=0.035$ that is significantly less than in the two other cases. This result indicates that the model equation of the second oscillator has the form of Eq. (2).

We use the smallest value of $L_{min}(\varepsilon,k)$ from the three obtained ones as a criterion for the identification of the right coupling. But if the minimal value of $L_{min}(\varepsilon,k)$ is close to the values of $L_{min}(\varepsilon,k)$ for the other two projections, it cannot be considered as the reliable criterion for identification of the coupling type. To ensure the validity of this criterion we use it only if the minimal value of $L_{min}(\varepsilon,k)$ is less than the other values of $L_{min}(\varepsilon,k)$ by a factor of 2 or a greater factor. The difference between the values of $L_{min}(\varepsilon,k)$ for different projections depends not only on the level of noise but also on the coupling coefficient. For small coupling values it is difficult to identify the *a priori* unknown type of coupling. For example, for the considered above parameter values of coupled Mackey-Class equations and the coupling coefficient $k_2=0.1$ we were able to identify with certainty the type of action of the system X_2 on the system X_1 for additive noise levels up to 20%. The limitations of the method in the presence of other types of noise need further investigation.

IV. CONCLUSION

We have proposed the method for estimation of coupling between two scalar time-delay systems of the form (1) based on the reconstruction of the model equations of coupled systems from their time series. The method is based on the statistical analysis of time intervals between extrema in the time series and the projection of the infinite-dimensional phase space of a time-delay system to suitably chosen two-dimensional subspaces. It can be successfully applied to time series under sufficiently high level of noise. The method can be used for the analysis of unidirectional and bidirectional coupling of time-delay systems and is able to estimate not only the coupling coefficients, but also the delay times, the nonlinear functions, and the parameters characterizing the inertial properties of coupled time-delay systems. It is shown that restricting consideration to several allowed types of coupling it is possible to estimate the coupling coefficients and to recover the coupled systems even in the case where the type of coupling between time-delay systems is *a priori* unknown. In this case the method allows one to identify the type of coupling.

The method can be also used for the reconstruction of a time-delay system affected by a system that is not a time-delay system and for the estimation of strength of this driving. In contrast to the other methods of detection of coupling between the systems from time series [17–19] the proposed technique is able to define not only the direction but also the value of coupling.

The method efficiency is illustrated using both numerical data, produced by coupled time-delay differential equations including the case of noise presence, and experimental data, gained from coupled electronic oscillators with delayed feedback.

The procedure of the coupling coefficients estimation considered with coupled time-delay systems like (2)–(4) for the three chosen types of linear coupling can be successfully applied to many other types of coupling between scalar time-delay systems of the form (1). For example, in the case of diffusive coupling between time-delay systems X_1 and X_2 described by the equation

$$\varepsilon_{1,2}\dot{x}_{1,2}(t) = -x_{1,2}(t) + f_{1,2}(x_{1,2}(t-\tau_{1,2})) + k_{2,1}(x_{2,1}(t) - x_{1,2}(t)), \qquad (10)$$

it is possible to recover the nonlinear functions $f_{1,2}$ and the parameters $\varepsilon_{1,2}$ and $k_{2,1}$ of the systems $X_{1,2}$ by plotting $\varepsilon_{1,2}\dot{x}_{1,2}(t)+x_{1,2}(t)-k_{2,1}(x_{2,1}(t)-x_{1,2}(t))$ versus $x_{1,2}(t-\tau_{1,2})$ under variation of ε and k. The method is also efficient for some types of nonlinear coupling between the systems X_1 and X_2 if the coupling term does not contain the unknown functions g_1 or g_2. In the case of the coupling term $k_{2,1}(g_{2,1}[x_{2,1}(t)]-g_{1,2}[x_{1,2}(t)])$ and other similar terms the method cannot be used. It should be noted that in the general case for the reconstruction of coupled time-delay systems and their coupling coefficients estimation we must know the type of coupling defining the embedding spaces to which the trajectories of time-delay systems are projected.

In principle, it is possible to extend the proposed method to time-delay systems described in the absence of coupling by delay-differential equation of higher order than Eq. (1):

$$\varepsilon_n x^{(n)}(t) + \varepsilon_{n-1}x^{(n-1)}(t) + \cdots + \varepsilon_1\dot{x}(t) = -x(t) + f(x(t-\tau)), \qquad (11)$$

where $x^{(n)}(t)$ is the time derivative of order n and $\varepsilon_1,\ldots,\varepsilon_n$ are the parameters characterizing the inertial properties of the system. However, the higher the order of equation, the more parameters of coupled systems have to be recovered. As a result, the time of computation significantly increases and the quality of reconstruction deteriorates since the procedure involves numerical calculation of the higher order derivatives. Similar problems arise in the case of three and more coupled time-delay systems.

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