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Equations of a Time-Delay System under External Action Reconstructed from Time Series

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Abstract—A new method is proposed for the use of time series for reconstructing the equations of a system with time-delay feedback under an external action. The performance of the proposed method is illustrated by application to short noisy time series of a model system under external actions of various types. © *2003 MAIK "Nauka/Interperiodica".*

Introduction. In recent years, the problem of reconstruction of the equations of nonlinear dynamical systems with time-delay feedback (time-delay systems) has received much attention [1–9]. The importance of this problem is related to the fact that time-delay systems are frequently encountered in the nature. The behavior of such systems is not entirely determined by the present state, but depends on the preceding states as well. Accordingly, the time-delay systems are usually described in terms of differential equations with delayed argument. Such models are successfully used in various fields of physics, biology, physiology, and chemistry. However, reconstruction of the model equations of a time-delay system from its time series in the case when this system occurs under the action of other system is still insufficiently studied, although this situation is encountered in solving many important practical problems.

In this paper, methods developed previously [6–8] for the reconstruction of the model equations of time-delay systems from their chaotic time series are extended so as to include such systems occurring under an external action.

Description of the method. Consider a time-delay system X described in the absence of external actions by a first-order differential equations with delayed argument of the following general type:

$$\varepsilon_0 \dot{x}(t) = -x(t) + f(x(t-\tau_0)), \tag{1}$$

where x is a dynamical variable describing the state of the system at the time t, f is a nonlinear function, τ_0 is the delay time, and ε_0 is a dimensionless parameter characterizing the inertia of the system. In the general case, Eq. (1) represents the mathematical model of an oscillatory system comprising a circuit with three ideal elements: nonlinear device, inertial element, and delay line. In Fig. 1, these elements of a circuit X are denoted by f, ε_0, and τ_0, respectively.

Now let another system Y to act upon system X. This action can be realized in different ways. We distinguish between three methods of coupling, by which a variable of system y is introduced with a certain coefficient into various points of circuit X (denoted by Roman numerals I–III in Fig. 1). Depending on the point of application of the system Y action upon system X, the dynamics of the latter system is described by one of the following equations:

$$\text{I: } \varepsilon_0 \dot{x}(t) = -x(t) + f(x(t-\tau_0) + k_y y(t-\tau_0)), \tag{2}$$

$$\text{II: } \varepsilon_0 \dot{x}(t) = -x(t) + f(x(t-\tau_0) + k_y y(t)), \tag{3}$$

$$\text{III: } \varepsilon_0 \dot{x}(t) = -x(t) + f(x(t-\tau_0) + k_y y(t)), \tag{4}$$

where $y(t)$ is a dynamical variable describing the state of system Y at the time t and k_y is the coupling coefficient characterizing the degree of the system Y action upon X.

The proposed method allows the time-delay system X to be reconstructed, the coupling mode to be determined (i.e., the situations described by Eqs. (1)–(3) to

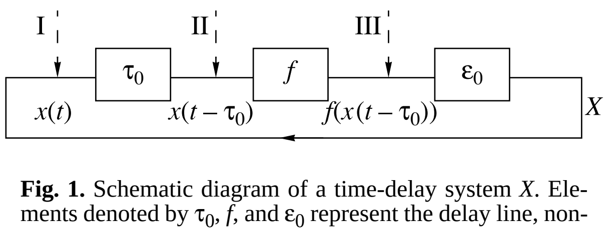

Fig. 1. Schematic diagram of a time-delay system X. Elements denoted by τ_0, f, and ε_0 represent the delay line, nonlinear device, and inertial transformation of oscillations in the system, respectively. Points I–III represent various modes of introduction of the external action (system Y) into system X.

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be distinguished), and the degree of external action (i.e., the value of k_y) to be estimated from the available time series of oscillations in systems Y and X. In order to determine the delay time τ_0 from the observed time series $x(t)$, we use the method developed in [6], where it was demonstrated that time series of the systems of type (1) contain virtually no extrema spaced from each other by τ_0. In order to find τ_0, we have to indicate extrema in the initial time series, determine the numbers N of the pairs of extrema spaced by various times τ, and construct the $N(\tau)$ dependence. Then, the delay time τ_0 is determined by the position of the absolute minimum of the $N(\tau)$ function. The results of our investigations showed that this approach can be also successfully used in cases when system X occurs under the action of another system Y, provided that this external action does not lead to the appearance of a large number of additional extrema in the time realizations of oscillations in system X.

For determining the parameter ε_0 and the function f of system X and the coupling coefficient k_y, we propose a method based on an analysis of the time series of both observables $x(t)$ and $y(t)$. First, let us assume that the mode of action of system Y upon system X (i.y., the structure of equation describing dynamics of the time-delay system under the external action) is known. For example, consider the coupling mode I described by Eq. (2), whereby the variable of system Y is introduced into the feedback circuit of X after the inertial element. As can be seen from Eq. (2), a manifold of points with the coordinates $(x(t-\tau_0) + k_y y(t-\tau_0), \varepsilon_0 \dot{x}(t) + x(t))$ plotted on the plane will reproduce the function f. Since the quantities ε_0 and k_y are not known a priori, we have to plot $\varepsilon \dot{x}(t) + x(t)$ versus $x(t-\tau_0) + ky(t-\tau_0)$ for various ε and k in search for the single-valued relationship that is possible only for $\varepsilon = \varepsilon_0$ and $k = k_y$. As a quantitative criterion of such a unique relationship in the search for ε_0 and k_y, we can use the minimum length of a segment $L(\varepsilon, k)$ connecting points (ordered with respect to abscissa) on the above plane. A minimum of $L(\varepsilon, k)$ will correspond to $\varepsilon = \varepsilon_0$ and $k = k_y$, while the dependence of $\varepsilon \dot{x}(t) + x(t)$ on $x(t-\tau_0) + ky(t-\tau_0)$ constructed for these parameters will reproduce a certain nonlinear function that can be approximated. The proposed approach employs all points of the time series, which allows short time series to be used for reconstruction of the system parameters ε_0 and k_y and the nonlinear function f.

The same method can be used for reconstructing the nonlinear function f and the parameters ε_0 and k_y in the situations described by Eqs. (3) and (4) by plotting $\varepsilon \dot{x}(t) + x(t)$ versus $x(t-\tau_0) + ky(t)$ and $\varepsilon \dot{x}(t) + x(t) - ky(t)$ versus $x(t-\tau_0)$, respectively, for various ε and k. If the point (I, II, or III) at which system Y acts upon system X is not known a priori, it is necessary to perform reconstruction for each of the three model equations (1)–(3). The only correct structure of the model equation will be indicated by single-valued form of the reconstructed function and, accordingly, by lowest of the three values of $L_{\min}(\varepsilon, k)$. Thus, the proposed method allows both the parameters of a time-delay system under external action and the mode of this action (i.e., the form of the model equation) to be reconstructed from the observed time series.

Verification of the method. We will demonstrate performance of the proposed reconstruction method by applying the procedure outlined above to a time-delay system X described by the Mackey–Glass equation,

$$\dot{x}(t) = -bx(t) + \frac{ax(t-\tau_0)}{1 + x^c(t-\tau_0)}, \tag{5}$$

under an external system Y producing a harmonic or chaotic action. Equation (5) reduces to the form of Eq. (1) with $\varepsilon_0 = 1/b$ and $f(x(t-\tau_0)) = ax(t-\tau_0)/b(1 + x^c(t-\tau_0))$.

Figure 2 shows the results of reconstruction of the Mackey–Glass type system in a chaotic regime ($a = 0.2$, $b = 0.1$, $c = 10$, $\tau_0 = 300$) under the action of system Y performing sinusoidal oscillations $y(t) = A\sin\omega t$ with an amplitude ($A = 1$) close to the amplitude of natural oscillations in system (5) and with the oscillation period $T = 2\pi/\omega = 130$. The type of Y–X coupling corresponds to mode I described by Eq. (2) with a coupling coefficient $k_y = 0.1$. The map of $N(\tau)$ (Fig. 2a) was constructed using a time series $\dot{x}(t)$ containing 5000 points. The derivative $x(t)$ was estimated from the time series by means of a local parabolic approximation. The absolute minimum of $N(\tau)$ allows the delay time to be exactly estimated as $\tau_0 = 300$.

Figures 2b–2d illustrate reconstruction of the nonlinear function for the ε and k values corresponding to a minimum of $L_{\min}(\varepsilon, k)$ for the model equation selected in the form (2), (3), or (4), respectively. These plots were constructed using only 1000 points for each of the $x(t)$ and $y(t)$ time series. In the search for $L_{\min}(\varepsilon, k)$, the parameter ε was varied at a step of 0.1 ($\varepsilon_0 = 1/b = 10$) and the parameter k, at a step of 0.01. For the model reconstructed in the form of Eq. (2), a minimum length of the $L(\varepsilon, k)$ segment normalized to the number of points was $L_{\min}(\varepsilon, k) = L(10.1, 0.10) = 0.007$. When the model was selected n the form of Eq. (3), we obtained $L_{\min}(\varepsilon, k) = L(7.4, -0.05) = 0.1r53$, while reconstruction in the form of Eq. (4) yielded $L_{\min}(\varepsilon, k) = L(7.3, 0.06) = 0.147$. Of the three plots only Fig. 2b shows a nearly single-valued relationship and the corresponding $L_{\min}(\varepsilon, k)$ value is significantly smaller than the two other. These results indicate that the model equation is correctly identified in the form (2) and shows that the parameters ε_0 and k_y are determined with a good accuracy.

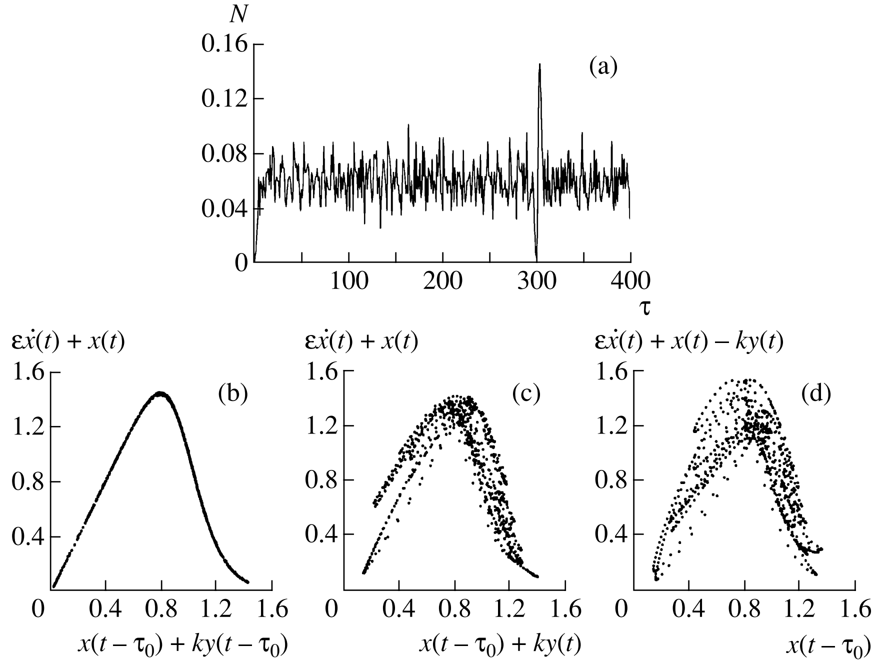

Fig. 2. Reconstruction of the Mackey–Glass system under external harmonic action: (a) plot of the number N of the pairs of extrema in a time series of X spaced by various times τ, normalized to the total number of such extrema ($N_{min}(\tau) = N(300)$; (b–d) reconstruction of the nonlinear function for a model equation selected in the form of Eqs. (2)–(4), respectively, with the parameters (b) ε = 10.1, k = 0.10; (c) ε = 7.41, k = –0.05; (d) ε = 7.3, k = –0.06.

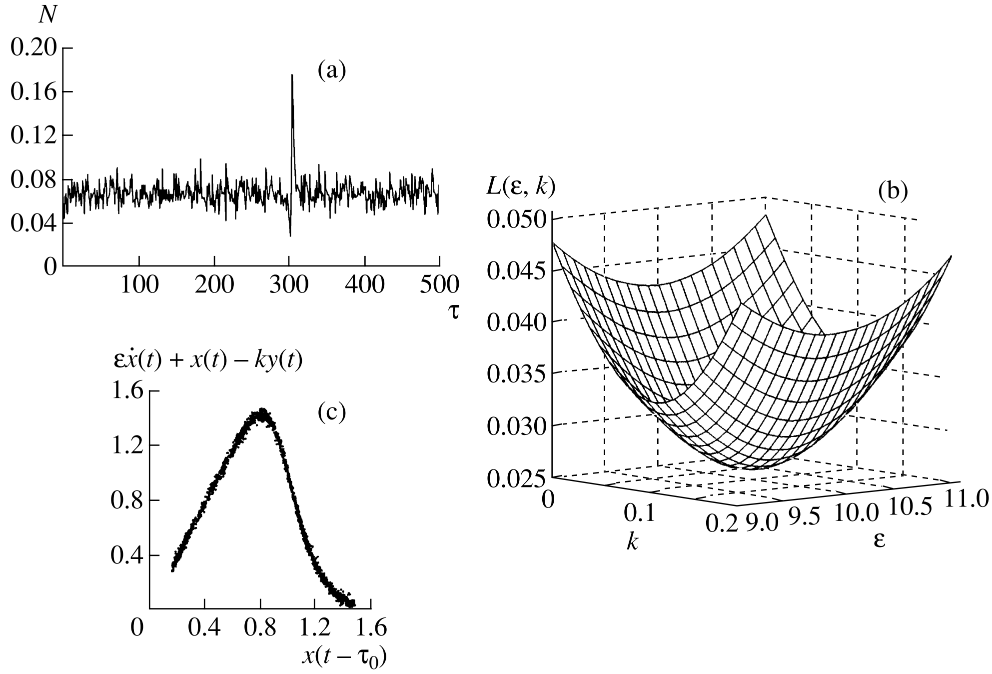

Fig. 3. Reconstruction of the Mackey–Glass system under the action of another Mackey–Glass system: (a) plot of the number N of the pairs of extrema in a time series of X spaced by various times τ, normalized to the total number of such extrema ($N_{min}(\tau)$ = N(300); (b) $L(\varepsilon, k)$ function for the model equation selected in the form of Eq. (4) normalized to the number of points ($L_{min}(\varepsilon, k)$ = L(10.1, 0.10) = 0.026); (c) reconstruction of the nonlinear function for ε = 10, k = 0.1.

For the parameters of systems X and Y indicated above, the proposed method allows the type of the model equation to be determined and system X to be reconstructed for $0.01 \le |k_y| \le 0.5$. In comparison to the other methods [10, 11] of determining the coupling between systems from their time series, out procedure has a number of advantages. In contrast to the method of directivity indices [11], the proposed procedure is applicable to synchronized systems and allows the magnitude of coupling (rather than only its direction) to be estimated even in the case of fundamentally different systems.

In order to check the performance of the proposed method in the presence of perturbations, we applied it to a system with noise. The procedure has proved to be more sensitive to noise in the time-delay system, but it still works when a noise level in system *X* is on the order of 10%. The level of noise in system *Y* can be several times greater than that in system *X*.

Finally, we have considered the case when the external action upon system *X* is produced by another time-delay system *Y*. Figure 3 shows the results of reconstruction for a Mackey–Glass type system with $a = 0.2$, $b = 0.1$, $c = 10$, $\tau_0 = 300$ in the presence of another Mackey–Glass type system with the same values of a, b, c, and $\tau_0 = 400$. The coupling between *Y* and *x* corresponded to mode III described by Eq. (4) with $k_y = 0.1$. In addition, both systems were perturbed by a Gaussian white noise with zero mean and an rms deviation amounting to 10% of that for the time series without noise. Despite the presence of noise the plot of $N(\tau)$ allows the delay time to be precisely estimated (Fig. 3a) and the $L(\varepsilon, k)$ map restores the values of ε_0 and k_y (Fig. 3b). The $L(\varepsilon, k)$ plot was constructed using 2000 points of $x(t)$ and $y(t)$ time series at a step of 0.1 for ε and 0.01 for k. The presence of noise significantly impairs the quality of reconstruction of the nonlinear function (Fig. 3c). For the model reconstructed in the form of Eq. (2), we obtain $L_{\min}(\varepsilon, k) = L(10.1, 0.01) = 0.042$, while reconstruction in the form of Eq. (3) gives $L_{\min}(\varepsilon, k) = L(10.0, -0.02) = 0.039$. This analysis indicates that the model equation should be identified in the form (4) that yields the minimum value of $L_{\min}(\varepsilon, k) = L(10.1, 0.10) = 0.026$.

Conclusion. We proposed a new method for reconstructing a time delay system under an external action from the observable time series. The method was verified for various types of external action and different ways of its introduction into the given system. The proposed method can be successfully applied to the analysis of short time series even at a rather high noise level.

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REFERENCES

1. M. J. Bünner, M. Popp, Th. Meyer, *et al.*, Phys. Rev. E: Stat. Phys., Plasmas, Fluids, Relat. Interdiscip. Top. **54**, 3082 (1996).
2. H. Voss and J. Kurths, Phys. Lett. A **234**, 336 (1997).
3. S. P. Ellner, B. E. Kendall, S. N. Wood, *et al.*, Physica D **110**, 182 (1997).
4. R. Hegger, M. J. Bünner, and H. Kantz, Phys. Rev. Lett. **81**, 558 (1998).
5. M. J. Bünner, M. Ciofini, A. Giaquinta, *et al.*, Eur. Phys. J. D **10**, 165 (2000).
6. A. S. Karavaev, V. I. Ponomarenko, and M. D. Prokhorov, Pis’ma Zh. Tekh. Fiz. **27** (10), 43 (2001) [Tech. Phys. Lett. **27**, 414 (2001)].
7. B. P. Bezruchko, A. S. Karavaev, V. I. Ponomarenko, and M. D. Prokhorov, Phys. Rev. E: Stat. Phys., Plasmas, Fluids, Relat. Interdiscip. Top. **64**, 056216 (2001).
8. V. I. Ponomarenko and M. D. Prokhorov, Pis’ma Zh. Tekh. Fiz. **28** (16), 37 (2002) [Tech. Phys. Lett. **28**, 680 (2002)].
9. V. S. Udaltsov, J.-P. Goedgebuer, L. Larger, *et al.*, Phys. Lett. A **308**, 54 (2003).
10. M. G. Rosenblum and A. S. Pikovsky, Phys. Rev. E: Stat. Phys., Plasmas, Fluids, Relat. Interdiscip. Top. **64**, 045202 (2001).
11. M. G. Rosenblum, L. Cimponeriu, A. Bezerianos, *et al.*, Phys. Rev. E: Stat. Phys., Plasmas, Fluids, Relat. Interdiscip. Top. **65**, 041909 (2002).

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