Recovery of Time-Delay Systems with Two Delays from Time Series

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We propose the method for reconstructing time-delay systems with two coexisting delay times from chaotic time series. The method is based on the characteristic location of extrema in time series of time-delay systems and the projection of infinite-dimensional phase space of these systems to suitably chosen low-dimensional subspaces. We verify our method by using it for the recovery of generalized Mackey-Glass time-delay differential equation from its chaotic time series.

Key words: parameter estimation, delay-differential equations, time series analysis

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1 Introduction

Systems, whose dynamics is affected not only by the current state, but also by past states, are wide spread in nature [1]. Usually these systems are modelled by delay-differential equations. Such models are successfully used in many scientific disciplines, like physics, physiology, biology, economics and cognitive sciences. Typical examples include population dynamics [2], where individuals participate in the reproduction of a species only after maturation, or spatially extended systems, where signals have to cover distances with finite velocities [3–5]. Within this rather broad class of systems, one can find the Mackey-Glass equation [6] modelling the production of red blood cells, and many other models in biosciences for different phenomena from glucose metabolism to infectious diseases [7]. In general, modelling the dynamics of time-delay systems it could be necessary to take into account the dependence of current state on several states in the past. In this case, the model with several delay times should be used [8–13].

To recover model equations of time-delay systems from time series several methods have been proposed recently [14–22]. However, practically all these methods have been applied for reconstruction of delay-differential equations with single delay time. Extension of these methods to time-delay systems with several coexisting delays is usually not possible. In this paper we develop the method of recovery of time-delay systems with two different delay times from chaotic time series. The paper is organized as follows. Section 2 contains the method description. In Section 3 the method efficiency is illustrated by the reconstruction of generalized Mackey-Glass equation from its chaotic time series. In Section 4 we summarize our results.

2 Method description

Let us consider a time-delay system with two different delay times τ_1 and τ_2

$$\dot{x}(t) = F\left(x(t), x(t-\tau_1), x(t-\tau_2)\right). \qquad (1)$$

To recover the delay times τ_1 and τ_2 from the temporal realization $x(t)$ we exploit the method proposed recently in [20], where we have shown that there are practically no extrema separated in time by τ in the time series of time-delay system $\dot{x}(t) = G(x(t), x(t-\tau))$. We will show that this method based on the characteristic location of extrema in the time series of time-delay systems can be successfully applied to time delay system (1) with two delays. Differentiation of Equation (1) with respect to t gives

$$\ddot{x}(t) = \frac{\partial F}{\partial x(t)}\dot{x}(t) + \frac{\partial F}{\partial x(t-\tau_1)}\dot{x}(t-\tau_1) + \frac{\partial F}{\partial x(t-\tau_2)}\dot{x}(t-\tau_2). \quad (2)$$

The realization $x(t)$ of Equation (1) has mainly quadratic extrema and therefore $\dot{x}(t) = 0$ and $\ddot{x}(t) \neq 0$ at the extremal points. Hence, if $\dot{x}(t) = 0$, the condition

$$a\dot{x}(t-\tau_1) + b\dot{x}(t-\tau_2) \neq 0 \quad (3)$$

must be fulfilled, where $a = \partial F(x(t), x(t-\tau_1), x(t-\tau_2))/\partial x(t-\tau_1)$ and $b = \partial F(x(t), x(t-\tau_1), x(t-\tau_2))/\partial x(t-\tau_2)$. The condition (3) can be satisfied only if $\dot{x}(t-\tau_1) \neq 0$ or/and $\dot{x}(t-\tau_2) \neq 0$. By this is meant that the derivatives $\dot{x}(t)$ and $\dot{x}(t-\tau_1)$, or $\dot{x}(t)$ and $\dot{x}(t-\tau_2)$ do not vanish simultaneously. As the result, the number of extrema separated in time by τ_1 and τ_2 from a quadratic extremum must be appreciably less than the number of extrema separated in time by other values of τ. Then, to define the delay times τ_1 and τ_2 one has to determine the extrema in the time series and after that to define for different values of time τ the number N of pairs of extrema separated in time by τ and to construct the $N(\tau)$ plot. The $N(\tau)$ plot will demonstrate pronounced minima at $\tau = \tau_1$ and $\tau = \tau_2$ corresponding to the delay times.

We illustrate the procedure for estimating the other characteristics of time-delay system with two delays from time series for the system governed by the following equation

$$\varepsilon_1\dot{x}(t) = -x(t) + f_1(x(t-\tau_1)) + f_2(x(t-\tau_2)), \quad (4)$$

where f_1 and f_2 are nonlinear functions and ε_1 is the parameter characterizing the inertial properties of the system. Time differentiation of Equation (4) gives

$$\varepsilon_1\ddot{x}(t) = -\dot{x}(t) + \frac{\partial f_1(x(t-\tau_1))}{\partial x(t-\tau_1)}\dot{x}(t-\tau_1) + \frac{\partial f_2(x(t-\tau_2))}{\partial x(t-\tau_2)}\dot{x}(t-\tau_2). \quad (5)$$

From Equation (5) it follows that if

$$\dot{x}(t-\tau_1) = \dot{x}(t-\tau_2) = 0, \quad (6)$$

then $\varepsilon_1\ddot{x}(t) = -\dot{x}(t)$ and

$$\varepsilon_1 = -\frac{\dot{x}(t)}{\ddot{x}(t)}. \quad (7)$$

Thus, to estimate the parameter ε_1 one can find the points of $x(t)$ satisfying condition (6), define for them the first and the second derivatives, calculate ε_1 using Equation (7), and conduct averaging.

To recover the nonlinear functions f_1 and f_2 we project the trajectory generated by Equation (4) to a three-dimensional space $(x(t-\tau_1), x(t-\tau_2), \varepsilon_1\dot{x}(t) + x(t))$. In this space the projected trajectory is confined to a two-dimensional surface since according to Equation (4)

$$\varepsilon_1\dot{x}(t) + x(t) = f_1(x(t-\tau_1)) + f_2(x(t-\tau_2)). \quad (8)$$

The section of this surface with the $x(t-\tau_2) =$ const plane enables one to recover the nonlinear function f_1 up to a constant since the points of the section are correlated via $\varepsilon_1\dot{x}(t) + x(t) = f_1(x(t-\tau_1)) + c_1$, where $c_1 = f_2(x(t-\tau_2))$ for some fixed value of $x(t-\tau_2)$. In a similar way one can recover up to a constant the nonlinear function f_2 by intersecting the trajectory projected to the above-mentioned three-dimensional space with the $x(t-\tau_1) = \text{const}$ plane. The points of this section are correlated via $\varepsilon_1\dot{x}(t) + x(t) = f_2(x(t-\tau_2)) + c_2$, where $c_2 = f_1(x(t-\tau_1))$ for fixed $x(t-\tau_1)$.

3 Method application

We demonstrate the method efficiency with a generalized Mackey-Glass equation obtained by introducing a further delay,

$$\dot{x}(t) = -bx(t) + \frac{a_1 x(t-\tau_1)}{2(1+x^c(t-\tau_1))} + \frac{a_2 x(t-\tau_2)}{2(1+x^c(t-\tau_2))}. \tag{9}$$

Division of Equation (9) by b reduces it to Equation (4) with $\varepsilon_1 = 1/b$. The parameters of the system (9) are chosen to be $a_1 = 0.3$, $a_2 = 0.2$, $b = 0.1$, $c = 10$, $\tau_1 = 70$ and $\tau_2 = 300$ to produce a dynamics on a high-dimensional chaotic attractor. For various τ values we count the number N of situations when $\dot{x}(t)$ and $\dot{x}(t-\tau)$ are simultaneously equal to zero and construct the $N(\tau)$ plot [Figure 1(a)]. The step of τ variation in Figure 1(a) is equal to unity. The time derivatives $\dot{x}(t)$ are estimated from the time series by applying a local parabolic approximation. The first two most pronounced minima of $N(\tau)$ are observed at $\tau_1' = 69$ and $\tau_2' = 300$. Another distinctive minimum of $N(\tau)$ is observed close to $\tau = \tau_1 + \tau_2$. Processing the points satisfying condition (6) with the recovered values τ_1' and τ_2' we obtain the averaged estimation $\varepsilon_1' = 9.4$ for the parameter $\varepsilon_1 = 1/b = 10$. To reduce inaccuracy in ε_1 determination by formula (7) we exclude from consideration the points with very small values of $\ddot{x}(t)$.

Projecting the time series of Equation (9) to the three-dimensional space $(x(t-\tau_1'), x(t-\tau_2'), \varepsilon_1'\dot{x}(t) + x(t))$ and constructing the sections of this space with the planes $x(t-\tau_2') = \text{const}$ and $x(t-\tau_1') = \text{const}$ we obtain at these sections the nonlinear functions f_1 and f_2 recovered up to a constant. However, inaccuracy in estimation of τ_1 and ε_1 leads to insufficient quality of the nonlinear function recovery.

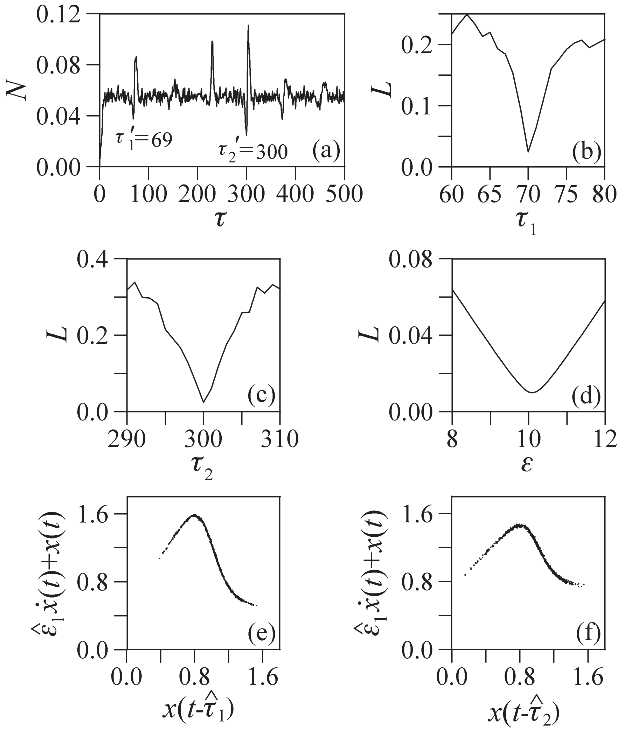

FIG. 1. (a) Number N of pairs of extrema in the time series of Equation (9) separated in time by τ, as a function of τ. $N(\tau)$ is normalized to the total number of extrema in the time series. (b) Length L of a line connecting points ordered with respect to abscissa in the $x(t-\tau_2') = 1$ section, as a function of τ_1. $L_{min}(\tau_1) = L(70)$. (c) Length L of a line connecting points ordered with respect to abscissa in the $x(t-\hat{\tau}_1) = 1$ section, as a function of τ_2. $L_{min}(\tau_2) = L(300)$. (d) Length L of a line connecting points ordered with respect to abscissa in the $x(t-\hat{\tau}_2) = 1$ section, as a function of ε. $L_{min}(\varepsilon) = L(10.1)$. (e) Nonlinear function f_1 recovered up to the constant $\hat{c}_1 = f_2(x(t-\hat{\tau}_2))$, where $x(t-\hat{\tau}_2) = 1$. (f) Nonlinear function f_2 recovered up to the constant $\hat{c}_2 = f_1(x(t-\hat{\tau}_1))$, where $x(t-\hat{\tau}_1) = 1$.

To achieve more high quality of the model equation reconstruction we propose the following procedure for the correction of the parameters. Varying τ_1 in a small vicinity of $\tau_1' = 69$ we project the time series to several three-dimensional $(x(t-\tau_1), x(t-\tau_2'), \varepsilon_1'\dot{x}(t) + x(t))$ spaces and plot their sections with the

$x(t-\tau_2') = \text{const}$ plane, searching for a section, which points contract to a curve demonstrating almost single-valued dependence. As a quantitative criterion of single-valuedness we use the minimal length of a line $L(\tau_1)$ connecting all points of the section ordered with respect to abscissa. The $L(\tau_1)$ plot demonstrates the minimum at $\hat{\tau}_1 = 70$ [Figure 1(b)]. Similarly, the correction of the delay time τ_2 is performed. We project the time series to $(x(t-\hat{\tau}_1), x(t-\tau_2), \varepsilon_1'\dot{x}(t) + x(t))$ spaces under variation of τ_2 in the vicinity of $\tau_2' = 300$ and plot the sections $x(t-\hat{\tau}_1) = \text{const}$. Note, that for these sections the corrected delay time $\hat{\tau}_1 = 70$ is used. The minimum of $L(\tau_2)$ takes place at $\hat{\tau}_2 = 300$ [Figure 1(c)]. In the general case if $\hat{\tau}_2 \neq \tau_2'$, the procedure of τ_1 revision is repeated by plotting the sections of the embedding spaces with the $x(t-\hat{\tau}_2) = \text{const}$ plane with the corrected delay time $\hat{\tau}_2$. Successive correction of τ_1 and τ_2 is continued until the parameters cease changing. For small deviations of initial estimates τ_1' and τ_2' from the true delay times the procedure is converging and allows one to define both delay times accurately.

After revision of the delay times the parameter ε_1 should be corrected. Its new estimate $\hat{\varepsilon}_1$ can be obtained by formula (7). However, a more reliable estimation is the one using all points of one of the section. To obtain it we project the time series to $(x(t-\hat{\tau}_1), x(t-\hat{\tau}_2), \varepsilon\dot{x}(t) + x(t))$ spaces under variation of ε in the vicinity of ε_1', searching for a single-valued dependence in the section $x(t-\hat{\tau}_1) = \text{const}$ or in the section $x(t-\hat{\tau}_2) = \text{const}$. The $L(\varepsilon)$ plot shows the minimum at $\hat{\varepsilon}_1 = 10.1$ [Figure 1(d)]. In Figure 1 the values of $L(\varepsilon)$, $L(\tau_1)$ and $L(\tau_2)$ are normalized to the number of points in the corresponding section. Note, that the proposed procedure of the successive correction of the parameters needs in several orders of magnitude smaller time of computation than the method of simultaneous selection of the parameters ε_1, τ_1 and τ_2 for the three-dimensional embedding space $(x(t-\tau_1), x(t-\tau_2), \varepsilon_1\dot{x}(t) + x(t))$.

Figures 1(e) and (f) illustrate the reconstructed nonlinear functions of the system with two coexisting delays (9) for the corrected parameters $\hat{\varepsilon}_1 = 10.1$, $\hat{\tau}_1 = 70$ and $\hat{\tau}_2 = 300$. The nonlinear functions f_1 and f_2 are recovered up to the constant by plotting the sections of the two-dimensional surface described by Equation (8). To investigate the method efficiency in the presence of noise we apply it to noisy data and found that the method provides sufficiently accurate reconstruction of the investigated system for noise levels up to 10%.

4 Conclusion

We have proposed the method for reconstructing time-delay systems with two coexisting delay times from chaotic time series. The method is based on the statistical analysis of time intervals between extrema in the time series and the projection of infinite-dimensional phase space of the time-delay system to suitably chosen low-dimensional subspaces. The procedure for the successive correction of the model equation parameters is proposed.

The method can be applied to the systems of different nature if these systems have similar structure of model equations. The proposed technique allows one to estimate the delay times, the parameter characterizing the inertial properties of the system and the nonlinear functions even in the case of noise presence. The method of the delay time definition uses only operations of comparing and adding. It needs neither ordering of data, nor calculation of approximation error or certain measure of complexity of the trajectory and therefore it does not need significant time of computation. The method efficiency is illustrated by the reconstruction of generalized Mackey-Glass equation from chaotic time series.

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References

[1] J. K. Hale, S. M. V. Lunel. *Introduction to Functional Differential Equations.* Springer. New York. (1993).

[2] Y. Kuang. *Delay Differential Equations With Applications in Population Dynamics.* Academic Press. Boston. (1993).

[3] K. Ikeda, K. Matsumoto. High-dimensional chaotic behavior in systems with time-delayed feedback. Physica D. **29**, 223-235 (1987).

[4] K. Ikeda. Multiple-valued stationary state and its instability of the transmitted light by a ring cavity system. Opt. Commun. **30**, 257-261 (1979).

[5] R. Lang, K. Kobayashi. External optical feedback effects on semiconductor injection lasers. IEEE J. Quantum Electron. **16**, 347-355 (1980).

[6] M. C. Mackey, L. Glass. Oscillations and chaos in physiological control systems. Science. **197**, 287-289 (1977).

[7] G. A. Bocharov, F. A. Rihan. Numerical modelling in biosciences using delay differential equations. J. Comp. Appl. Math. **125**, 183-199 (2000).

[8] K. L. Cooke, P. van den Driessche. Analysis of an SEIRS epidemic model with two delays. J. Math. Biol. **35**, 240-260 (1996).

[9] S. A. Gourley, M. V. Bartucelli. Parameter domains for instability of uniform states in systems with many delays. J. Math. Biol. **35**, 843-867 (1997).

[10] H. Seidel, H. Herzel. Bifurcations in a nonlinear model of the baroreceptor–cardiac reflex. Physica D. **115**, 145-160 (1998).

[11] N. Burič, N. Vasovič. Oscillations in an excitable system with time-delays. Int. J. of Bifurcation and Chaos. **13**, 3483-3488 (2003).

[12] Y. Jiang, S.-H. Dong, M. Lozada-Cassou. Noise-induced coherence in bistable systems with multiple time delays. Phys. Rev. E. **69**, 056225 (2004).

[13] Y. Song, M. Han, Y. Peng. Stability and Hopf bifurcations in a competitive Lotka–Volterra system with two delays. Chaos, Solitons and Fractals. **22**, 1139-1148 (2004).

[14] M. J. Bünner, M. Popp, Th. Meyer, A. Kittel, J. Parisi. Tool to recover scalar time-delay systems from experimental time series. Phys. Rev. E. **54**, 3082-3085 (1996).

[15] H. Voss, J. Kurths. Reconstruction of non-linear time delay models from data by the use of optimal transformations. Phys. Lett. A. **234**, 336-344 (1997).

[16] S. P. Ellner, B. E. Kendall, S. N. Wood, E. McCauley, C. J. Briggs. Inferring mechanism from time-series data: delay differential equations. Physica D. **110**, 182-194 (1997).

[17] M. J. Bünner, Th. Meyer, A. Kittel, J. Parisi. Recovery of the time-evolution equation of time-delay systems from time series. Phys. Rev. E. **56**, 5083-5089 (1997).

[18] R. Hegger, M. J. Bünner, H. Kantz, A. Giaquinta. Identifying and modeling delay feedback systems. Phys. Rev. Lett. **81**, 558-561 (1998).

[19] M. J. Bünner, M. Ciofini, A. Giaquinta, R. Hegger, H. Kantz, R. Meucci, A. Politi. Reconstruction of systems with delayed feedback: (I) Theory. Eur. Phys. J. D. **10**, 165-176 (2000).

[20] B. P. Bezruchko, A. S. Karavaev, V. I. Ponomarenko, M. D. Prokhorov. Reconstruction of time-delay systems from chaotic time series. Phys. Rev. E. **64**, 056216 (2001).

[21] V. I. Ponomarenko, M. D. Prokhorov Extracting information masked by the chaotic signal of a time-delay system. Phys. Rev. E. **66**, 026215 (2002).

[22] V. S. Udaltsov, J.-P. Goedgebuer, L. Larger, J.-B. Cuenot, P. Levy, W. T. Rhodes. Cracking chaos-based encryption systems ruled by nonlinear time delay differential equations. Phys. Lett. A. **308**, 54-60 (2003).