Reconstruction of time-delay systems from chaotic time series

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We propose a method that allows one to estimate the parameters of model scalar time-delay differential equations from time series. The method is based on a statistical analysis of time intervals between extrema in the time series. We verify our method by using it for the reconstruction of time-delay differential equations from their chaotic solutions and for modeling experimental systems with delay-induced dynamics from their chaotic time series.

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I. INTRODUCTION

Reconstruction of model equations from time series with the help of universal methods, ignoring object features is, as a rule, unsuccessful. Usually, a good result can be expected when special techniques of reconstruction are used for sufficiently narrow classes of objects. In particular, such a class can be composed of objects, whose dynamics is affected not only by the present state, but also by past states. These systems are usually modeled by delay-differential equations. Such models are successfully used in many disciplines, like physics, biology, and chemistry. Some of them, for example, the Mackey-Glass equation [1], the Ikeda equation [2], and equation for an electronic oscillator with delayed feedback [3] became standard examples of time-delay systems. Our paper deals with the problem of the time-delay system reconstruction from experimental chaotic time series. We consider one of the most popular first-order delay-differential equations

$$\varepsilon\dot{x}(t)=-x(t)+f(x(t-\tau_0)), \qquad (1)$$

where $x(t)$ is the system state at time t, function f defines nonlocal correlations in time, τ_0 is the delay time, and parameter ε characterizes the inertial properties of the system. In general case Eq. (1) is a mathematical model of an oscillating system composed of a ring with three ideal elements: nonlinear, delay, and inertial. In a radiophysical version of the ring (Fig. 1), which is named an electronic oscillator with delayed feedback, an amplifier with the transfer function f plays the role of nonlinear device, a delay line provides a delay for time τ_0, and a filter defines the system inertial properties and the parameter ε. In the present paper we develop a technique for estimating τ_0, f, and ε from the time series.

To uniquely define the system (1) state it is necessary to prescribe the initial conditions in the entire time interval $[-\tau_0,0]$. Therefore, the phase space of the system has to be considered as infinite dimensional. In fact, for large delay times even scalar delay-differential equations can possess high-dimensional chaotic dynamics [4]. Thus, the direct reconstruction of the system by the time-delay embedding techniques runs into severe problems. For a successful recovery of the time-delay systems one has to use special methods. For example, in the method proposed in [5,6] the trajectory generated by Eq. (1) is projected from the infinite-dimensional phase space to a three-dimensional space $[x(t-\tau_0),x(t),\dot{x}(t)]$. In this space the projected trajectory is confined to a two-dimensional surface. The section of this surface with the $\dot{x}(t)=0$ plane enables one to recover the nonlinear function since when $\dot{x}(t)=0$, then

$$x(t)=f(x(t-\tau_0)). \qquad (2)$$

Since the delay time τ_0 is *a priori* unknown, one needs to project the trajectory to several $[x(t-\tau),x(t),\dot{x}(t)]$ spaces upon variation of τ searching for a single-valued dependence in the $\dot{x}(t)=0$ section, which is possible only for $\tau=\tau_0$. As a quantitative criterion of single valuedness in searching for τ_0 one can use the minimal length $L(\tau)$ of a line connecting all extreme points ordered with respect to $x(t-\tau)$ in the $[x(t-\tau),x(t)]$ plane [6]. Other methods of time-delay system analysis based on the similar projection of the infinite-dimensional phase space onto low-dimensional subspaces use another criteria of quality, for example, the minimal forecast error of constructed model [7–9], the minimal value of information entropy [10], or various measures of complexity of the projected time series [7,11,12]. Several methods of time-delay system analysis exploit regression analysis [13,14] and correlation function construction [15,16].

In this paper we propose a method that is able to reconstruct the equation of time-delay system having the form of Eq. (1) from the time series. The method uses regularities in the location of extrema in the system (1) time series. Section II contains the method description. We determine extrema in

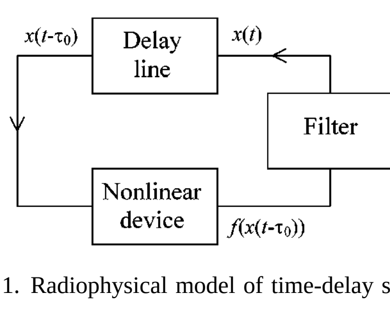

FIG. 1. Radiophysical model of time-delay system.

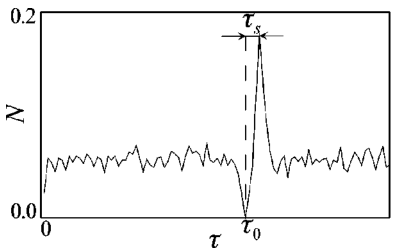

FIG. 2. Number N of pairs of extrema in a realization of Eq. (1) with $\varepsilon > 0$, separated in time by τ, as a function of τ. $N(\tau)$ is normalized to the total number of extrema in time series. $N(\tau)$ has a pronounced minimum at the level of the delay time of the system. The location of maximum is determined by the parameter ε.

the time series and analyze the time intervals between them. For different values of time τ we define the number N of pairs of extrema separated in time by τ (Fig. 2). The characteristic features of the $N(\tau)$ plot allows us to identify the delay time τ_0 without calculation of any additional quantitative criteria. With a knowledge of τ_0, it is possible to estimate the nonlinear function and the parameter ε. The value of ε can be estimated directly from the $N(\tau)$ plot. In Sec. III the method features and efficiency in the presence of noise are illustrated both by the reconstruction of dynamical systems from their solutions and by the modeling of real radiophysical systems. The method advantages and its possible applications are discussed in Sec. IV.

II. METHOD DESCRIPTION

The proposed method exploits the features of extrema shape and location in the system (1) temporal realization $x(t)$. The peculiarities of extrema location in time are clearly illustrated by $N(\tau)$ plot in Fig. 2. To construct it one has to define for different τ values the number N of pairs of extrema in $x(t)$, that are separated in time by τ. If N is normalized to the total number of extrema, then for sufficiently large extrema number it can be used as an estimation of probability to find a pair of extrema in $x(t)$ separated by the interval τ. Let us explain the qualitative features of $N(\tau)$ for various values of parameter ε.

In the absence of inertial properties ($\varepsilon = 0$) Eq. (1) takes the form of Eq. (2). Its time differentiation gives

$$\dot{x}(t) = \frac{df(x(t-\tau_0))}{dx(t-\tau_0)}\dot{x}(t-\tau_0). \qquad (3)$$

From Eq. (3) it follows that if $\dot{x}(t-\tau_0) = 0$, then $\dot{x}(t) = 0$. Thus, for $\varepsilon = 0$ every extremum of $x(t)$ is followed within the time τ_0 by the extremum.[1] As the result, $N(\tau)$ shows a maximum for $\tau = \tau_0$ in Fig. 3(a).

[1]For chaotic temporal realizations of the systems under investigation practically all critical points with $\dot{x}(t) = 0$ are the extremal ones, and therefore we call the points with $\dot{x}(t) = 0$ the extremal points throughout this paper.

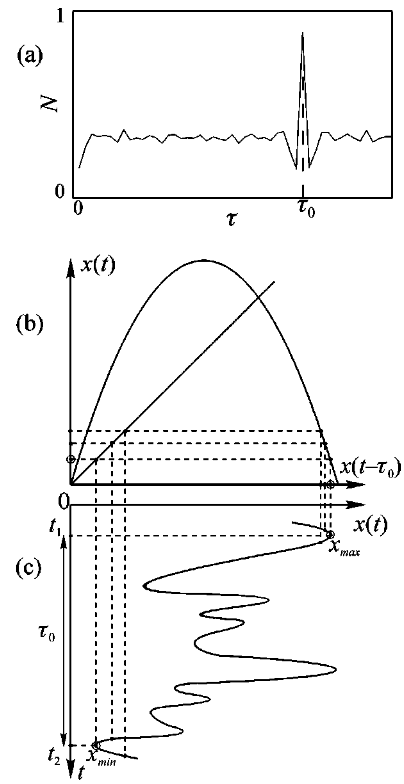

FIG. 3. (a) Number N of pairs of extrema in a realization of Eq. (1) with $\varepsilon = 0$, separated in time by τ, as a function of τ. $N(\tau)$ is normalized to the total number of extrema in time series. $N(\tau)$ has a sharp maximum at the level of the delay time of the system. (b) Typical transfer function of the nonlinear device and mapping of input signal points into output. (c) Signal temporal realization with the time series points (dots) shown in the neighborhood of two extremal points (circles).

The situation in the absence of inertial properties can be pictorially shown with the help of a ring circuit (Fig. 1), for which the condition $\varepsilon = 0$ is equivalent to the lack of filter and the unbounded passband of other elements. The signal $x(t)$ propagates through the ring in one direction and in the process the delay line provides the signal delay for τ_0 and a nonlinear device transforms the signal in accordance with its transfer function $f(x(t-\tau_0))$. In this case the signal at the nonlinear device output is defined at the time t only by the signal at the delay line input at the time $t-\tau_0$. Hence, the time evolution of the points of $x(t)$ can be represented by the iteration diagram of the one-dimensional map $x(t-\tau_0) \rightarrow x(t)$ in Fig. 3(b), where one step of discrete time corresponds to the time shift τ_0 in the continuous time. Graphical plotting of the mapping of several neighbor points chosen in $x(t)$ in the neighborhood of extremum [Fig. 3(c)] indicates that an extremum always maps into the extremum. From Fig. 3 it follows that the number of extrema separated in time by τ slightly differing from τ_0 must be relatively small resulting in the presence of minima in Fig. 3(a). In actuality we have to deal not with the continuous $x(t)$ realization but with a discrete time series $\{x_t\}_{t=1}^{M}$ obtained as a result of numerical solution of differential equation or experimental measurement of the system state x at the discrete time points. How-

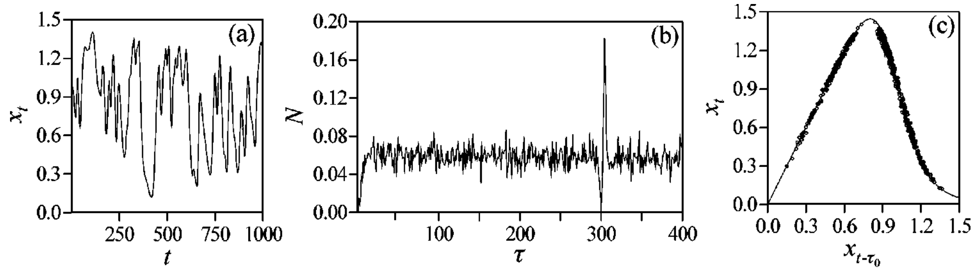

FIG. 4. (a) 1000 points of a realization of Eq. (6) with $a=0.2$, $b=0.1$, $c=10$, $\tau_0=300$. (b) Normalized number N of pairs of extrema in the time series separated in time by τ for $\tau=1,\ldots,400$. $N(\tau)$ is normalized to the total number of extrema in the time series. (c) Comparison of the function (7) (solid line) with its recovery from the time series (circles).

ever, as can be seen from Fig. 3(c), in this case the situation is also typical, when an extreme point of the time series is followed by the extremum within the time τ_0.

In the presence of inertial properties ($\varepsilon>0$), which corresponds to real situations, the most probable value of the time interval between extrema in $x(t)$ shifts from τ_0 to larger values. This effect can be explained using the ring system shown in Fig. 1, the filter introduces a certain additional delay in the system. As the result, the extrema in $x(t)$ can be found most often at the distance τ_0 plus τ_s apart (Fig. 2). For instance, the computational investigation of Eq. (1) with quadratic nonlinear function $f(x)=\lambda-x^2$ allows us to obtain an estimation $\tau_s\approx\varepsilon/2$ for large values of the parameter of nonlinearity λ.

For $\varepsilon>0$ the extrema in $x(t)$ are close to quadratic ones and therefore $\dot{x}(t)=0$ and $\ddot{x}(t)\neq 0$ at the extremal points. It can be shown that in this case there are practically no extrema in $x(t)$ separated in time by τ_0. To prove this let us differentiate Eq. (1) with respect to t:

$$\varepsilon\ddot{x}(t)=-\dot{x}(t)+\frac{df(x(t-\tau_0))}{dx(t-\tau_0)}\dot{x}(t-\tau_0). \tag{4}$$

If for $\dot{x}(t)=0$ in a typical case $\ddot{x}(t)\neq 0$, then, as it can be seen from Eq. (4), for $\varepsilon\neq 0$ the condition $\dot{x}(t-\tau_0)\neq 0$ must be fulfilled. Thus, there must be no extremum separated in time by τ_0 from a quadratic extremum and, hence, $N(\tau_0)\to 0$. For $\tau\neq\tau_0$, the derivatives $\dot{x}(t)$ and $\dot{x}(t-\tau)$ can be simultaneously equal to zero, i.e., it is possible to find extrema separated in time by τ. The specific configuration presented in Fig. 2 in the neighborhood of $\tau=\tau_0$ is duplicated at larger τ in the neighborhood of $\tau=2\tau_0, 3\tau_0, \ldots$.

The shape of $N(\tau)$ plot constructed from finite time series $\{x_t\}_{t=1}^{M}$ depends on length of the time series, sampling rate, noise level, and measurement accuracy. The dependence of these parameters on $N(\tau)$ shape and on the quality of model equation reconstruction is examined in Sec. III.

On the basis of the dependence of N on τ the following approach can be proposed to estimate the parameters of the time-delay model of form (1) from the time series:

(i) First of all, one has to determine the extrema in the time series. Then, for different τ it is necessary to define the number N of situations for which the time series points x_t and $x_{t-\tau}$ are simultaneously the extremal ones and to construct the $N(\tau)$ plot. The absolute minimum of $N(\tau)$ located near the absolute maximum is observed at the delay time τ_0.

(ii) After determination of the delay time τ_0 it is possible to reconstruct the nonlinear function $f(x(t-\tau_0))$ by plotting the extremal points x_t versus $x_{t-\tau_0}$. According to Eq. (2), the constructed set of points reproduces the unknown nonlinear function, which can be approximated if necessary.

(iii) The parameter ε can be estimated using the recovered nonlinear function, since from Eq. (1)

$$\varepsilon=\frac{f(x(t-\tau_0))-x(t)}{\dot{x}(t)}. \tag{5}$$

It is advisable to determine ε using all time series points, for which $\dot{x}_t\neq 0$ and the function $f(x_{t-\tau_0})$ is defined, and then to conduct averaging. The value of ε can be estimated directly from the $N(\tau)$ plot if the relation between ε and τ_s is known.

The proposed method of τ_0 determination does not need significant time of computation because only operations of comparing and adding can be used for the extrema definition and $N(\tau)$ construction.

III. METHOD APPLICATION

To test the efficiency of the proposed technique we have used it to reconstruct the equations of time-delay systems of the form (1) from the time series gained from their numerical solution and to model real ring oscillators from experimental time series. All time series used throughout this paper have 10 000 points. The time derivatives $\dot{x}_t$ were estimated from the time series by applying a local parabolic approximation.

A. Reconstruction of the Mackey-Glass equation

We apply the method to a time series produced by the Mackey-Glass equation

$$\dot{x}(t)=-bx(t)+\frac{ax(t-\tau_0)}{1+x^c(t-\tau_0)}, \tag{6}$$

which can be converted to Eq. (1) with $\varepsilon=1/b$ and the function

$$f(x(t-\tau_0))=\frac{ax(t-\tau_0)}{b[1+x^c(t-\tau_0)]}. \tag{7}$$

With the help of a fourth-order Runge-Kutta method the time series is calculated for the parameters producing a dynamics on a high-dimensional chaotic attractor [4]. Part of the time series is shown in Fig. 4(a). The time series is sampled in such a way that 300 points in time series cover a period of time equal to the delay time $\tau_0=300$. The time series exhibits about 600 extrema. Figure 4(b) illustrates the τ dependence of the number N of pairs of extrema separated in time by τ. In this figure as well as in the subsequent figures N is normalized to the total number of extrema in the time series. The absolute minimum of $N(\tau)$ takes place exactly at $\tau=\tau_0=300$, where $N(300)=0$. As the length M of the time series (and, hence, the number of extrema) decreases, this minimum in the $N(\tau)$ plot becomes less pronounced and at $M<2500$, when the time series exhibits about 150 extrema, additional minima appear with $N(\tau)=0$. In Fig. 4(c) we compare the true model function with its recovery from the time series. The estimated value of ε, averaged over all time series points, for which $\dot{x}_t\neq 0$ and $f(x_{t-\tau_0})$ is defined, is $\varepsilon=10.6$ (its true value is $\varepsilon=1/b=10$).

To investigate the robustness of the method to additional noise we analyze the data produced by adding to the time series of Eq. (6) zero-mean Gaussian white noise with a standard deviation of 3% and 10% of the standard deviation of the data without noise. The presence of noise in time series brings into existence spurious extrema. These extrema are not caused by the intrinsic dynamics of a system and temporal distances between them are random. With the extrema number increasing, a probability to find a pair of extrema in time series separated in time by τ has to increase in general. As a result, with noise increasing the average N value in Figs. 5(a) and 5(c) becomes greater. The extrema number increasing induced by noise is also followed by the increase of probability to find a pair of extrema separated by the interval τ_0. However, for moderate noise levels this probability is still less than the probability to find a pair of extrema separated in time by $\tau\neq\tau_0$. For instance, for noise level of 3% $N_{min}(\tau)=N(300)=0.02$ in Fig. 5(a) and for noise level of 10% $N_{min}(\tau)=N(300)=0.07$ in Fig. 5(c). For higher noise levels the absolute minimum of $N(\tau)$ is no longer observed for $\tau=\tau_0$. Since the absolute minimum of $N(\tau)$ is very well pronounced in the absence of noise, it can be clearly distinguished even in the noise presence if the noise level is not very high. Hence, the qualitative features of the $N(\tau)$ plot specified by the delay-induced dynamics are retained for a moderate noise level.

The presence of noise is more critical for the nonlinear function recovery. As the noise level increases, the set of points in the $(x_{t-\tau_0},x_t)$ plane becomes more dispersed [Figs. 5(b) and 5(d)]. To smooth the time series corrupted by noise and to reduce the number of extrema caused by noise one can use more nearest-neighbor points in the procedure of local approximation while estimating derivatives from data. Such approach allows us to observe a more pronounced minimum of $N(\tau)$ than that shown in Fig. 5(c) for the Mackey-Glass equation with a noise level of 10%. For sufficiently high levels of noise the absolute minimum of $N(\tau)$ can be sometimes distinguished for $\tau=\tau_0$ if one increases the length of the considered time series.

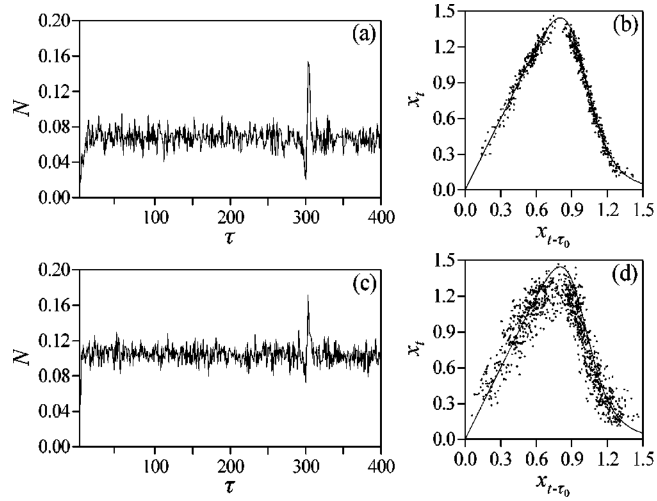

FIG. 5. Delay estimation and function reconstruction from the time series of the Mackey-Glass equation with additive Gaussian white noise. (a),(c) Normalized number N of pairs of extrema in the time series separated in time by τ for noise levels of 3% (a) and 10% (c). $N(\tau)$ is normalized to the total number of extrema in the time series. (b),(d) Nonlinear function (7) (solid line) and the estimated functions (dots) for noise levels of 3% (b) and 10% (d).

B. Modeling of electronic oscillator with delayed feedback

In Fig. 6(a) the block diagram of the electronic oscillator with delayed feedback is sketched for the case when the filter

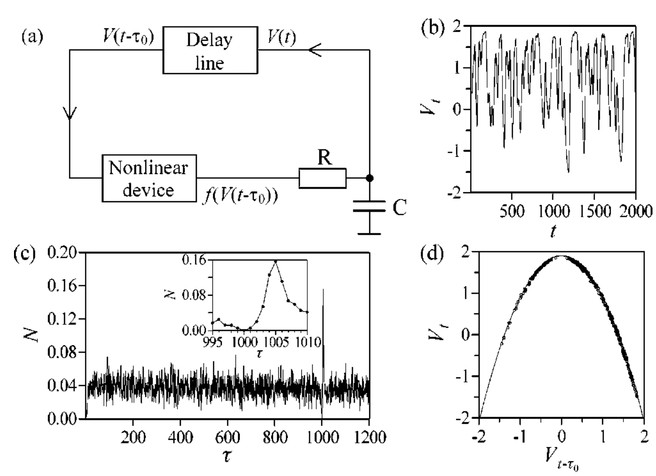

FIG. 6. (a) Block diagram of the electronic oscillator with delayed feedback. (b) 2000 points of a realization of Eq. (8) with the nonlinear function (9) for $\lambda=1.9$, $\tau_0=1000$, $RC=10$. (c) Number N of pairs of extrema in the time series separated in time by τ normalized to the total number of extrema. The inset shows a fragment of $N(\tau)$ for $\tau=995,\ldots,1010$. $N_{min}(\tau)=N(1000)=0$. (d) Comparison of the function (9) (solid line) with the recovered function (circles).

is a low-frequency first-order RC filter. We used several versions of this scheme with various combinations of analogue and digital elements that were connected with the help of analog-to-digital and digital-to-analog converters. We apply the method to a time series of voltage V across the filter capacitor. From Kirchhoff's laws, one can derive the model equation of such system

$$RC\dot{V}(t) = -V(t) + f(V(t-\tau_0)), \tag{8}$$

where $V(t)$ and $V(t-\tau_0)$ are the delay line input and output voltages, respectively; R and C are the resistance and capacitance, respectively. Eq. (8) is of form (1) with $\varepsilon = RC$.

Figure 6(b) shows part of a realization of Eq. (8) with the following nonlinear function

$$f(V) = \lambda - V^2, \tag{9}$$

where λ is a nonlinearity parameter. The time series is sampled in such a way that 1000 points in time series cover a period of time equal to the delay time $\tau_0 = 1000$. The time series exhibits about 400 extrema. $N(\tau)$ presented in Fig. 6(c) allows us to define τ_0 accurately. The true nonlinear function and the recovered function are compared in Fig. 6(d). The estimated from the time series $\varepsilon = RC = 9.9$. It is possible to estimate RC from the magnitude $\tau_s = \tau_m - \tau_0$, where τ_m is the value, at which the absolute maximum of $N(\tau)$ is observed. By varying the values of RC, λ, and τ_0 within a wide range, we obtained the following empirical relationship: $\tau_s \approx RC/2 = \varepsilon/2$. Thus, one can approximately estimate RC directly from $N(\tau)$ using the relationship $RC \approx 2\tau_s$. Note, that such estimate may be more accurate than the others in the presence of noise, when the recovered nonlinear function is not single valued and has to be averaged. Since the maximum of $N(\tau)$ is clearly defined for two to three times higher noise levels than the $N(\tau)$ minimum, the value of τ_m can be used as an upper estimate of τ_0 from the data heavily corrupted by noise.

In Fig. 7 we apply the method to two experimental time series produced by a setup [Fig. 6(a)] with radiophysical RC filter. In Figs. 7(a) and 7(b) the nonlinear device and the delay line are simulated on the computer and in Figs. 7(c) and 7(d) these elements are the radiophysical ones as well as the filter. The delay time is accurately identified in Fig. 7(a) from the time series sampled with a time step, which is ten times smaller than the parameter $\varepsilon = RC$ characterizing the system inertial properties. The correct definition of the delay time is verified by closeness of the recovered nonlinear function in Fig. 7(b) to a single-valued dependence. For a sampling interval comparable to ε, $N(\tau)$ does not fall to zero (even in the absence of noise) and has an additional minimum at the right of maximum [Fig. 7(c)]. The delay time τ estimated by the location of $N(\tau)$ minimum adjacent to the maximum at the left, is smaller than the true τ_0 by the value of sampling time. As a result, the recovered nonlinear function in Fig. 7(d) is not single valued. Therefore, $N(\tau)$ having qualitatively the shape as in Fig. 7(c), indicates the necessity to increase the sampling rate while recording the time series.

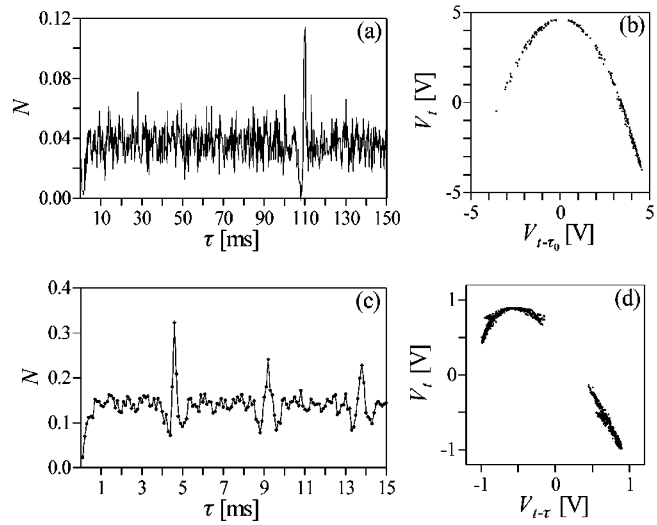

FIG. 7. (a),(c) Number N of pairs of extrema in the experimental time series separated in time by τ normalized to the total number of extrema. Sampling interval is much smaller than ε (a) and is comparable to ε (c). $\tau_0 = 108$ ms, $N_{min}(\tau) = N(108\ \text{ms}) = 0$ (a); $\tau_0 = 4.5$ ms, $N(4.5\ \text{ms}) = 0.18$, $N(4.4\ \text{ms}) = 0.07$ (c). (b),(d) Estimated nonlinear functions for relatively small (b) and relatively large (d) sampling interval. $\tau = 4.4$ ms (d).

IV. CONCLUSION

We have proposed the method of reconstruction of scalar time-delay differential equations having the form of Eq. (1) from time series. The method is based on the characteristic location of quadratic extrema in the time series and the statistical analysis of time intervals between them. If the system has inertial properties, the quadratic shape of extrema is typical and in practice one can analyze all extrema in the time series. The absolute minimum of $N(\tau)$ plot allows one to estimate the delay time τ_0, which can be used after that for the nonlinear function reconstruction and estimation of ε. Since the maximum of $N(\tau)$ is more pronounced than the $N(\tau)$ minimum and is clearly distinguished for two to three times higher noise levels than the minimum, its location can be used independently as an upper estimate of τ_0 from the data heavily corrupted by noise.

The proposed method of the delay time definition uses only operations of comparing and adding. It needs neither ordering of data, nor calculation of approximation error or certain measure of complexity of the trajectory and therefore it does not need significant time of computation. Thus, the method is perspective for data analysis in the real time with the help of compact computing devices. The method application can be useful for the development of techniques alternative to those proposed in [9,17] for solving the problem of extraction of messages masked by chaotic signals of time-delay systems.

The method efficiency is illustrated by the reconstruction of time-delay differential equations from the time series produced by these equations including the case of noise presence and by modeling experimental time-delay systems. The procedure of the delay time estimation from the $N(\tau)$ plot considered with systems such as Eq. (1) can be successfully applied to time series gained from a more general class of time-delay systems

$$\dot{x}(t)=F(x(t),x(t-\tau_0)). \tag{10}$$

Time differentiation of Eq. (10) gives

$$\ddot{x}(t)=\frac{\partial F(x(t),x(t-\tau_0))}{\partial x(t)}\dot{x}(t)+\frac{\partial F(x(t),x(t-\tau_0))}{\partial x(t-\tau_0)}\dot{x}(t-\tau_0). \tag{11}$$

Similar to Eq. (4), Eq. (11) implies that in the case of quadratic extrema derivatives $\dot{x}(t)$ and $\dot{x}(t-\tau_0)$ do not vanish simultaneously, i.e., if $\dot{x}(t)=0$, then $\dot{x}(t-\tau_0)\neq 0$.

In principle, it is possible to extend the proposed method of τ_0 definition from time series to high-dimensional time-delay systems having the following form:

$$x^{(n)}(t)+a_1x^{(n-1)}(t)+\cdots+a_{n-1}\dot{x}(t)=F(x(t),x(t-\tau_0)), \tag{12}$$

where $x^{(n)}(t)$ is the derivative of order n and $a_1,\ldots,a_{n-1}$ are the coefficients. Differentiation of Eq. (12) with respect to t gives

$$x^{(n+1)}(t)+a_1x^{(n)}(t)+\cdots+a_{n-1}\ddot{x}(t)=\frac{\partial F(x(t),x(t-\tau_0))}{\partial x(t)}\dot{x}(t)+\frac{\partial F(x(t),x(t-\tau_0))}{\partial x(t-\tau_0)}\dot{x}(t-\tau_0). \tag{13}$$

The condition $\dot{x}(t-\tau_0)\neq 0$ for $\dot{x}(t)=0$ will be satisfied if the left-hand side of Eq. (13) does not vanish. In general, a probability to obtain zero in the left-hand side of Eq. (13) is very small and therefore, the $N(\tau)$ plot qualitatively must have a shape similar to that inherent in the case of first-order delay-differential equations such as Eqs. (1) and (10).

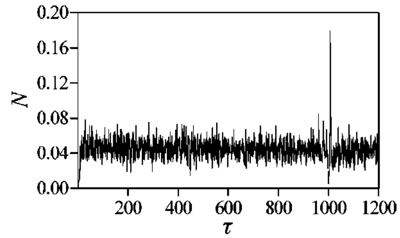

FIG. 8. Normalized number N of pairs of extrema in a realization of Eq. (14) with $\tau_0=1000$, $\lambda=1.9$, $\varepsilon=5$, separated in time by τ. $N(\tau)$ is normalized to the total number of extrema in the time series. $N_{min}(\tau)=N(1000)=0.006$.

To verify the method efficiency for a system of form (12) we have applied it to a time series gained from an electronic oscillator with delayed feedback that is similar to that shown in Fig. 6(a), but contains two identical in-series RC filters. The model equation for this oscillator with a two-section filter derived from Kirchhoff's laws has the form of second-order delay-differential equation

$$\varepsilon^2\ddot{V}(t)+2\varepsilon\dot{V}(t)=-V(t)+f(V(t-\tau_0)), \tag{14}$$

where $V(t)$ and $V(t-\tau_0)$ are the delay line input and output voltages, respectively, $\varepsilon=RC$. Figure 8 illustrates the $N(\tau)$ plot constructed for a case, where the nonlinear function in Eq. (14) has the form of Eq. (9). The absolute minimum of $N(\tau)$ allows us to define τ_0 accurately. Thus, the proposed technique of the delay time definition from time series can be successfully applied to a wide class of time-delay systems.

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