Technical Physics Letters, Vol. 26, No. 11, 2000, pp. 982–985. Translated from Pis'ma v Zhurnal Tekhnicheskoĭ Fiziki, Vol. 26, No. 22, 2000, pp. 7–14.
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Probability Symmetry Broken upon Rapid Period-Doubling Bifurcation

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Received May 23, 2000

Abstract—A quadratic one-dimensional mapping with noise is used to model the phenomenon of violation of the equal probability of variants in the postbifurcation system evolution upon a fast change of the control parameter. The case of a period-doubling bifurcation, after which only one of the two types of oscillations with different phases may occur in the system, is analyzed. The laws of the phenomenon are considered and its possible mechanism is discussed. © *2000 MAIK "Nauka/Interperiodica".*

1. Dealing with the nonlinear world, one frequently encounters bifurcation situations, whereby the initial state (motion) of a system loses stability in response to a change in the control parameter, replaced by two equiprobable variants of the subsequent existence and development pathways. In particular, a period-doubling bifurcation leads to the appearance of a cycle with a double period (in the vicinity of the cycle in which the system has lost stability), with two equiprobable variants of the subsequent motion differing only in phase (i.e., in the timeshift per period of the unstable motion). Then, the question naturally arises as to which of the two variants of the postbifurcation development will be realized in the system provided that the control parameter r varies at a certain finite rate S in the presence of a noise with the variance σ^2. The interest in answering this question is related both to studying fundamental problems in physics and biology (see, e.g., reviews [1, 2]) and to solving numerous practical tasks.

In the case where the control parameter varies very slowly (adiabatically) after the period-doubling bifurcation, the system should occur in the vicinity of the cycle where stability was lost and where selection of the subsequent state is determined by the noise. If the noise probability distribution is symmetric, the final states of the system are equiprobable as well: $p_1 = p_2 = 1/2$. In the opposite limit of infinitesimal noise ($\sigma^2 \longrightarrow 0$), either the first ($p_1 = 1, p_2 = 0$) or the second ($p_1 = 0, p_2 = 1$) final state is realized with certainty, depending on the initial conditions and irrespective of the rate S, which corresponds to completely predictable behavior. These opposite limiting cases of the bifurcation transition are called the "stochastic" and "dynamic" variants, respectively [3, 4].

Should the control parameter vary at a sufficiently high rate and the noise be present, an intermediate case is realized, whereby the final states can only be predicted at a probability lower than unity and the probability distribution is no longer symmetric ($p_1 \neq p_2$). In this case, it is a common practice to establish a conditional boundary between the stochastic and dynamic variants of the bifurcation transition, based on the criterion of attaining a certain final state at a preset probability (e.g., $p_1 = 0.75$; $p_2 = 0.25$). Butkovskiĭ *et al.* [3] studied, using numerical methods, the properties of a logistic mapping of several types [see Eqs. (2) below] with additive noise. It was established that the noise level and the rate of the control parameter variation corresponding to this boundary are related by a universal power law:

$$\sigma_c^2 = CS^{\alpha}. \tag{1}$$

Here, the exponent α and the coefficient C depend on the initial conditions, α being maximum when the initial coordinate x_0 is set at the boundaries of the "attraction domain" of the final state [4].[1]

The aim of this study was to elucidate a mechanism responsible for the violation of the final state probability symmetry in systems subject to a period-doubling bifurcation upon rapid change in the control parameter, to determine the role of the universal relationship (1), and to refine the concept of a fast transition. For this purpose, we have considered a greater number of models as compared to that used in the previous studies and gained a markedly greater volume of statistical data.

2. A classical model system with period-doubling bifurcations is offered by a one-dimensional mapping of the type $x_{n+1} = f(x_n)$, where $f(x_n)$ is a function with quadratic maximum, x is a dynamic variable, and $n = 0$,

[1] By the "attraction domain" we imply a part of the attraction basin of the double-period cycle, from which the system falls into one of the final states with different phases.

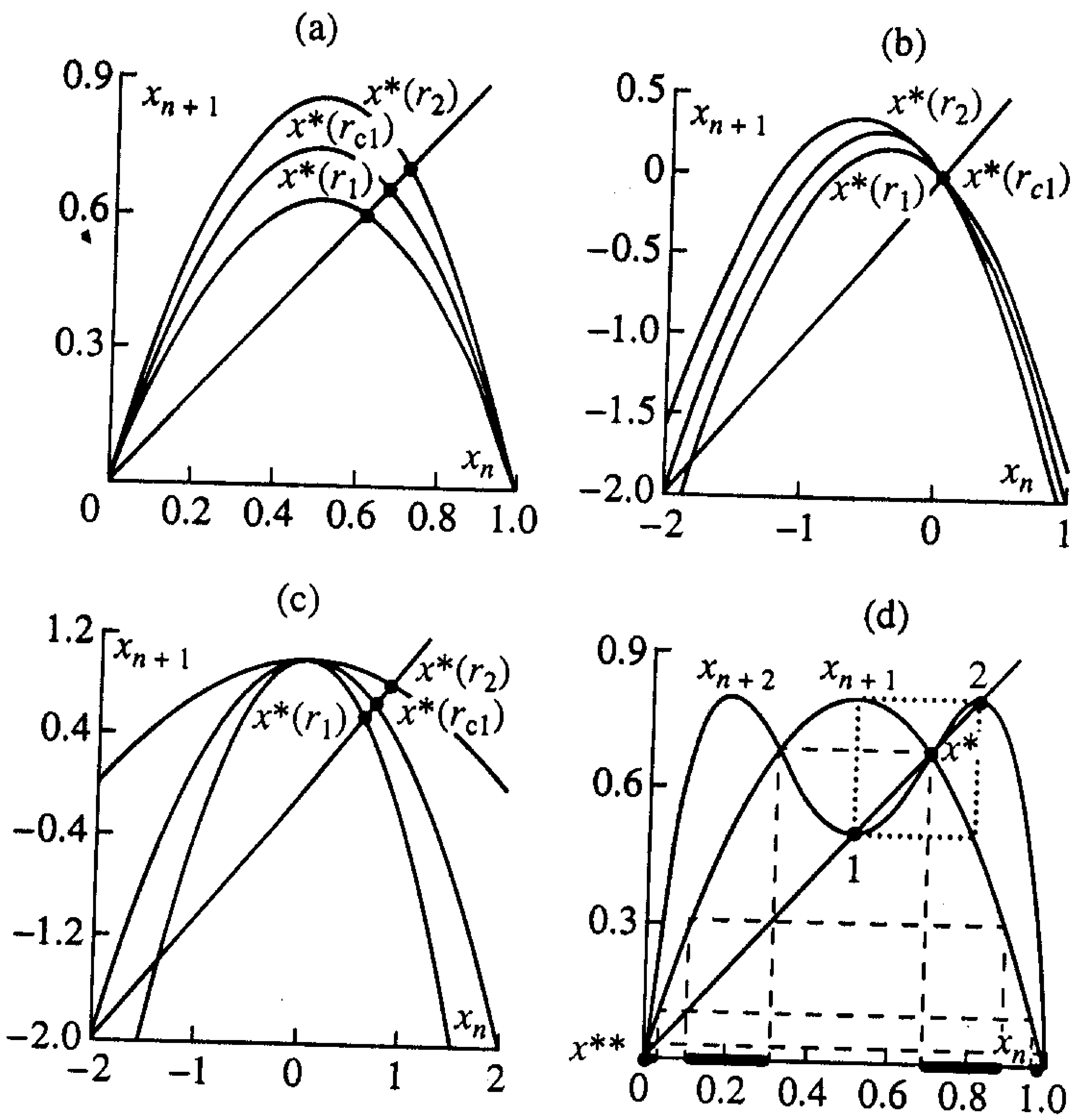

Fig. 1. The shapes of various mapping functions for different values of the control parameters: (a) mapping (2); (b) mapping (4); (c) mapping (6); (d) the structure of the attraction domains of states *1* (thin portions of the abscissa axis) and *2* (thick segments) for mapping (2) and its second iteration $x_{n+2} = f(f(x_n))$.

1, 2, … is the discrete time. We have studied five mapping functions of this type:

$$x_{n+1} = rx_n(1-x_n), \tag{2}$$

$$x_{n+1} = r - x_n^2, \tag{3}$$

$$x_{n+1} = -(x_n/r)(r+x_n), \tag{4}$$

$$x_{n+1} = r(1-r^3x_n^2), \tag{5}$$

$$x_{n+1} = 1 - rx_n^2. \tag{6}$$

These functions, which can be converted one into another by the corresponding substitution of variables, differ by the character of dependence of their stationary solutions x^* (equilibrium point, cycle of period 1) on the parameter r. In Fig. 1, these solutions are represented by intersections with the diagonal $x_n = x_{n+1}$.[2] The mapping function (4) has a zero derivative ($\gamma = dx^*/dr = 0$), which implies that point x^* does not change its position in the one-dimensional phase space (Fig. 1b). For the other functions, the derivatives are nonzero ($\gamma \neq 0$) and may differ in both magnitude and sign. In this case, the points x^* change their positions depending on the parameter (Figs 1a and 1c).

[2] The other point of intersection with the diagonal corresponds to an unstable solution x^{**}; this point separates the domains of finite and infinite solutions.

The control parameter was varied as described by a piecewise linear function:

$$r(n) = \begin{cases} r_1 + Sn, & n \leq N, \\ r_1 + SN = r_2, & n > N', \end{cases} \tag{7}$$

where r_1 and r_2 are the initial and final values of the control parameter and S is the parameter variation rate (determined as $S = \Delta r/N$, where $\Delta r = r_2 - r_1$ is the change in the control parameter and N is the number of iterations corresponding to this change). The bifurcation (critical) value of the parameter r_c, for which the multiplier is $|(dx_{n+1}/dx_n)|_{x^*} = 1$, falls within the interval $r_1 < r_c < r_2$. At $r = r_c$, the point (state) x^* loses stability and a cycle of period 2 appears in the vicinity of this point. In Fig. 1d showing mapping (2) and the corresponding second iteration $x_{n+2} = f(f(x_n))$, the double-period cycle corresponds to the equilibrium points 1 and 2. For a dynamic system with $x_{n+2} = f(f(x_n))$, these points appear as attractors with the attraction basins indicated by thick solid segments on the abscissa axis. For system (2), the different parts of the abscissa axis represent the "attraction domains" of the final states.

The noise was introduced into the mapping functions in the form of a small additive $x_n \longrightarrow x_n + \xi_n$, where ξ_n is a random sequence with a distribution law close to the normal (Gaussian) with a zero mean value. In the calculations, the initial condition was taken in the

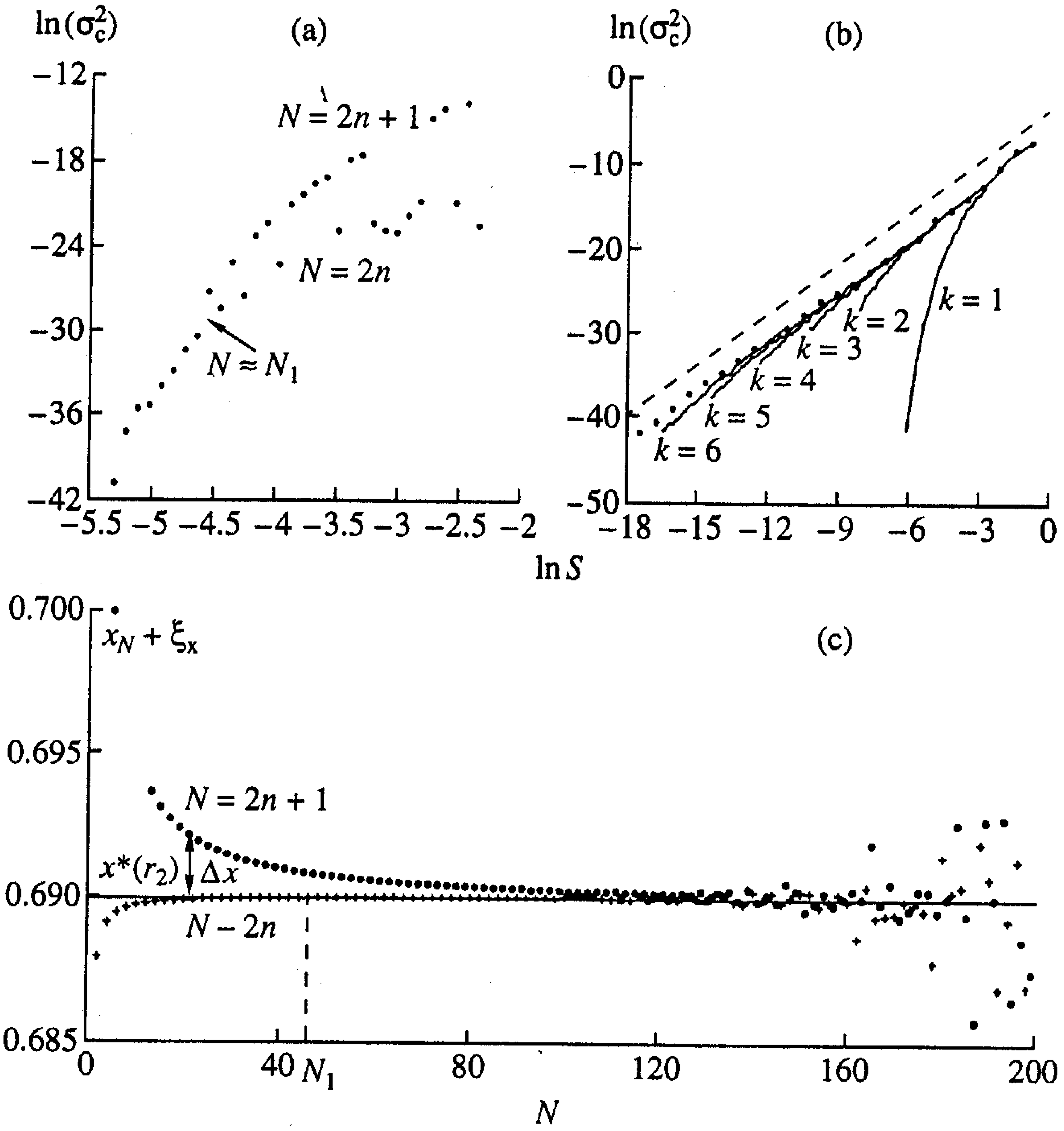

Fig. 2. Behavior of the system modeled by mapping (2): (a) a logarithmic plot of the critical noise variance versus the control parameter variation rate for a large Δr; (b) logarithmic plots of the critical noise variance versus the control parameter variation rate for an odd N and various Δr, dashed line shows analytical estimates calculated by formula (8); (c) the plots of $(x_N + \xi_N)$ and x^* for $r = r_2$ versus the number of iteration steps at a fixed Δr value.

form $x_0 = x^*(r_1)$ (Figs. 1a–1c) and determined analytically using Eqs. (2)–(6). Then the iteration procedure was conducted including a number of steps ($n \gg N$) that were sufficiently large for the transient processes to decay, after which the form of the final state was determined. Using the results of 250 numerical experiments, the probabilities $p_{1,2}$ for a given double-period state to be attained were calculated as a function of the noise variance σ^2 for various values of S. The smoothened $p_{1,2}(\ln S)$ plots were used to determine the critical noise level σ_c^2 for which the probabilities $p_{1,2}$ were equal to the preset values 0.75 and 0.25. By these estimates, the plots of $\ln(\sigma_c^2)$ versus $\ln S$ were constructed.

3. An analysis of the results of our numerical experiments leads to the following conclusions:

(a) In the general case (arbitrary Δr), dependence of the critical noise dispersion σ_c^2 on the parameter variation rate S differs from the power law and is determined by the type of mapping. This is illustrated by Fig. 2a showing that the plot of $\ln(\sigma_c^2)$ versus $\ln S$ exhibits two branches corresponding to odd and even N, none of which can be approximated by a linear function. Moreover, this plots depends on the Δr value as is seen from Fig. 2b (presenting the plots for odd N), where various Δr at the same S correspond to different σ_c^2.

(b) The power relationship (1) is valid only for small changes of the control parameter Δr, which is seen in Fig. 2b showing that portions of the plots with large k values can be approximated by straight lines. A large k corresponds to small $\Delta r_k = 2^{-3(k-1)}\Delta r_1$ (Δr_1 was taken to be equal to a difference of the control parameter values for the first and second period-doubling bifurcations). Note that the points corresponding to small N ($N = 1$ on the right-hand ends of the plots and $N = 200$ on their left-hand ends) also fit virtually to the same straight line.

(c) The above features and the very fact of violation of the final state probability symmetry in the systems studied is related to a shift of the "center of gravity" of the distribution function at the end of the fast transition (for $\langle x_N + \xi_N \rangle$, where angular brackets denote the ensemble-average) with respect to the boundary between the attraction domains of different final states. These boundaries are formed by unstable equilibrium points x^* and their images as illustrated in Fig. 1d (for details, see [5]). Over the interval $n = (0, N)$ (corresponding to r changing from r_1 to r_2), the x^* value varies monotonically with time, while x_n exhibits oscillations with a double quasi-period. Fluctuations, which are suppressed in the first stage of the transition (for $r < r_c$), grow in the second stage. Accordingly, the dispersion $\langle((x_N + \xi_N) - \langle x_N + \xi_N\rangle)^2\rangle$, which is not large for small N values, increases with N and the fluctuation component eventually becomes dominating. As is seen in Fig. 2c, there is a certain N_1 below which the quantity $(x_N + \xi_N)$ on each subsequent iteration step falls on different sides of the boundary x^*. This makes the different final states alternatively more probable (which results in separation of the points corresponding to odd and even N values in the right-hand part of Fig. 2a). For $N > N_1$, where the oscillating $(x_N + \xi_N)$ values fall on one side of the boundary, the parity of N no longer affects the σ_c^2 versus S curve. As the N value increases further, the development of fluctuations in the second stage of the transition increases the random character of the process, as reflected by the leveling of the probability of different final states.

The shift of $\langle x_N + \xi_N\rangle$ from boundary toward the domain of attraction of one of the possible final states (making this state more probable) depends both on the type of mapping and on the fast transition parameters. In particular, for mapping (4) with the derivative $\gamma = 0$, the position of the boundary x^* between the attraction domains of the final states is independent of r; for $x_0 = x^*$, the value of $\langle x_N + \xi_N\rangle = x^*$ also remains constant. Here, the symmetrically distributed noise equiprobably "drives" the system toward both possible evolution pathways for any N. In the general case of a mapping with $\gamma \neq 0$, including mappings of the types (2), (3), (5), and (6), the shift $\langle x_N + \xi_N\rangle$ is nonzero and the final state probability symmetry is broken.

(d) The dashed line in Fig. 2b shows estimates of the σ_c^2 values for various S obtained in the region of small Δr assuming that $x^*(r)$ is a linear function of r (i.e., $\gamma =$ const). For $N = 1$, the shift is $\langle x_N + \xi_N\rangle - x^* = \Delta x = S\gamma$. Then the probability of a given final state for a normal (Gaussian) noise with a zero mean is

$$p_1 = \int_{-\infty}^{\Delta x} \rho(\xi)d\xi = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{S\gamma}{\sigma\sqrt{2}}\right)\right], \quad (8)$$

where $\rho(\xi)$ is the noise probability density function. The error function for the preset critical value $p_1 = 0.75$ (used to determine the final variant) is $\mathrm{erf}(S\gamma/(\sigma_c 2^{1/2})) = 1/2$, which corresponds to $S\gamma/(\sigma_c 2^{1/2}) \approx 0.477$. As seen in Fig. 2b, the slopes of the calculated plots of $\ln(\sigma_c^2)$ versus $\ln S$ corresponding to small Δr values, as well as the variation of the positions of points corresponding to $N = 1$, agree with the line of estimates.

4. The phenomenon of violation of the final state probability symmetry upon the fast period-doubling bifurcation in the discrete models studied is determined by the boundaries of the attraction domains of different final states, which move in the phase space in response to variation of the control parameter. The levels of noise corresponding to the conditional boundary separating the "stochastic" and "dynamic" variants of the bifurcation transition are determined by properties of the model mapping. In the general case, the power relationship between the noise level and the control parameter variation rate predicted in [3] is valid only in a small vicinity of the critical (bifurcation) value.

Acknowledgments. The work was supported by the Russian Foundation for Basic Research, project nos. 99-02-17735 and 00-02-17441.

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Translated by P. Pozdeev