Constructing nonautonomous differential equations from experimental time series

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An approach to constructing model differential equations of harmonically driven systems is proposed. It is a modification of the standard global reconstruction technique: an algebraic polynomial which coefficients depend on time is used for approximation. Efficiency and details of the approach are demonstrated by various numerical and natural examples.

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I. INTRODUCTION

Different methods of reconstructing ordinary differential equations (ODE's) from time series have been developed for the last two decades [1–4]. They were applied successfully to solve the problems of prediction [5], qualitative description of dynamics [5,6], calculation of dimensions and Lyapunov exponents of an attractor [7], and signal classification [8,9]. The success of modeling from a time series depends essentially on the choice of a model structure—a number of model equations and a kind of basis functions used for approximation. This choice is the most difficult part of a reconstruction procedure and it can hardly be completely reduced to an algorithm. The methods employed involve, as a rule, reconstruction of ODE's in *a universal form*. Thus, a widely used approach [4–7,10] which we call *standard* involves reconstruction of equations in the form

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ &\cdots, \\ \dot{y}_D &= f(y_1, y_2, \ldots, y_D), \end{aligned} \tag{1}$$

where y_1 is an observable, f is a polynomial of an order K:

$$f(y_1, y_2, \ldots, y_D) = \sum_{l_1, l_2, \ldots, l_D = 0}^{K} c_{l_1, l_2, \ldots, l_D} \prod_{j=1}^{D} y_j^{l_j}, \quad \sum_{j=1}^{D} l_j \leqslant K. \tag{2}$$

However, such methods appear to be inefficient in many situations. In particular, universal models often contain a large number of coefficients and demonstrate divergent solutions. These difficulties may arise, e.g., when nonautonomous systems which this paper is devoted to are modeled.

A modification of the standard approach in application to modeling objects of a sufficiently small class seems to be a promising way of further development of the global reconstruction methods. Such a modification was used in application to harmonically driven oscillators in [11–13], where harmonically excited electric oscillatory circuits with nonlinear capacitance were modeled. The modification consisted in reconstruction of *nonautonomous equations*, that is, explicit functions of time were incorporated in model equations:

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= f(y_1, y_2) + a \cos \omega t + b \sin \omega t, \end{aligned} \tag{3}$$

where f is a polynomial. The model structure (3) reflects only the case of *additive* harmonic driving.

In this paper a general approach to global nonlinear modeling of harmonically driven systems is proposed. It can be effective not only for additive, but for an arbitrary way of harmonic driving. This approach is more general due to another modification of the standard model structure. It is assumed that all model coefficients may depend on time, i.e., the coefficients $c_{l_1, l_2, \ldots, l_D}$ in (2) are replaced by the expressions $(c_{l_1, l_2, \ldots, l_D} + a_{l_1, l_2, \ldots, l_D} \cos \omega t + b_{l_1, l_2, \ldots, l_D} \sin \omega t)$.

This approach is described in details in Sec. II B which follows the discussion of the possibilities and limitations of the standard method (Sec. II A). Efficiency of the new modification is demonstrated in Sec. III where harmonically driven (in various ways) systems are modeled. Advantages and potentialities of the approach are discussed in Sec. IV.

II. MODELING NONAUTONOMOUS SYSTEMS

A. The standard approach

Let us consider, in brief, the standard approach to the global reconstruction and discuss possible reasons of its inefficiency in many cases. The problem is as follows. Experimental data are in the form of a scalar time series, that is a finite sequence of values of an observable v: $\{v_i\}_{i=1}^{N_v}$, $v_i = v(t_i)$, $t_i = i\Delta t$, $i = 1, \ldots, N_v$, where Δt is a sampling interval. The realization observed is sufficiently smooth. It is necessary to construct model ODE's that demonstrate a behavior qualitatively similar to the original one and allow for an accurate short-term prediction. Under the standard approach, a procedure for solving the problem follows. First, a certain value of a model dimension D is selected. Second, state vectors $\mathbf{y}(t_i)$ are reconstructed by the method

of sequential derivatives (D time derivatives are obtained by differentiating the observed series numerically [5,10]). Third, the dependency $\dot{y}_D(\mathbf{y})$ is approximated by a polynomial (2). The values of coefficients $c_{l_1,l_2,\ldots,l_D}$ are calculated by the least squares technique. That is, their values are selected so as to minimize

$$\varepsilon^2=\frac{1}{N}\sum_{i=1}^{N}(\dot{y}_D(t_i)-f(y_1(t_i),y_2(t_i),\ldots,y_D(t_i)))^2, \quad (4)$$

where N is a number of state vectors reconstructed from the scalar time series. Now, the problem is reduced to solving a system of linear algebraic equations since the function f is linear with respect to the fitting coefficients. Finally, efficiency of a model obtained is checked. If the model is not effective then one should change (as a rule, increase) a polynomial order K or a model dimension D until a satisfactory model is found.

Experience of using the above procedure shows that the standard structure of model equations (1) and (2) does not allow for effective description of an observed process at all if models of moderate dimensions (approximately 3–4) do not work well. Reconstructed equations often demonstrate a behavior qualitatively different from the original one, in particular, divergent solutions. There are several reasons for this. First, increase of the model dimension D (that means increase of the order of numerical differentiation) results in increase of the effect of noise. Second, a polynomial of a high order may appear to be necessary for approximation. If the values of D and K are big, then a standard model contains a very large number of coefficients M: $M=(D+K)!/D!K!$. The more is the value of M, the less accurate are the estimates of the coefficients. Third, an orbit in a model phase space may go out of the region containing experimental data due to the approximation error. Then, the behavior of the model is no longer connected with the behavior of the object.

To summarize, one of the main reasons of the standard approach inefficiency is the big size of the models. This difficulty is, in general, unavoidable because the standard structure of model equations (1) and (2) is not, as a rule, a ''natural'' and the most relevant one for the variety of real systems and situations. Moreover, employing the standard model structure results in increase of a model dimension. It can be seen clearly by considering the reconstruction of dynamical systems.[1] As it follows from the results of Takens [15] and Sauer, Yorke, and Casdagli [16], any system of ODE's can be written in the form (1) for almost every observable v. But the value of D should satisfy the condition $D>2d$, where d is the dimension of a smooth manifold within which an observed motion occurs or the box-counting dimension of an attractor (if the motion corresponds to an attractor).

Thus, one can assume the direction of refining universal models. It consists in decreasing a model dimension and a number of its coefficients due to the change of a model structure. This change should lean on knowledge of specific features of an object under investigation, universality of models being in part lost.

B. The modification of model equations structure

So, a possible way of the standard approach development is the choice of a model structure (in application to a certain class of objects) that would allow for obtaining efficient models of *less dimension*. Such a choice can be done if one knows *characteristic features* of objects which belong to this class. In this paper we propose a special structure for modeling harmonically driven systems. A model takes the form

$$\begin{aligned}\dot{y}_1&=y_2,\\ \dot{y}_2&=y_3,\\ &\cdots,\\ \dot{y}_D&=f(y_1,y_2,\ldots,y_D,t),\end{aligned} \quad (5)$$

where $y_1=v$ and the function f depends on time t explicitly. It is easy to show that f can depend on time only via functions $\cos\omega t$ and $\sin\omega t$ in this case. We have employed a polynomial of an order K with varying coefficients:

$$\begin{aligned}f(y_1,y_2,\ldots,y_D,t)&\\ =\sum_{l_1,l_2,\ldots,l_D=0}^{K}&(c_{l_1,l_2,\ldots,l_D}+a_{l_1,l_2,\ldots,l_D}\cos\omega t\\ &+b_{l_1,l_2,\ldots,l_D}\sin\omega t)\prod_{j=1}^{D}y_j^{l_j}, \quad \sum_{j=1}^{D}l_j\leqslant K. \quad (6)\end{aligned}$$

So, f is linear with respect to the terms $\cos\omega t$ and $\sin\omega t$. In general, these harmonic functions can enter the expression for f in an arbitrary way, but (6) is a simple and an effective approximation for a sufficiently wide class of systems (see examples in Sec. III). A more sophisticated method of taking the driving terms into account is also possible, e.g., one can use higher powers of $\cos\omega t$ and $\sin\omega t$ along with their first powers. But the size of a model can increase significantly, followed by the above-mentioned difficulties.

One does not need to develop a new algorithm for the construction of a model (5) and (6). It is possible to employ the standard procedure (Sec. II A) if the value of a driving period T is preliminary calculated from a time series. The latter is necessary for application of the linear least squares routine to determining the other model coefficients. A way of finding the value of a period was suggested in [11–13] for the construction of a model (3). It can also be used in our case without any change. The idea is as follows. First, a sufficiently big value of a polynomial order K is selected. Second, an initial estimate $T=T^*$ of the period value is found (this estimate can be derived as a location of a peak in the power spectrum of the observed series). At this value of

[1]We use the term ''dynamical system'' to denote a mathematical object only, in contrast to the existing practice of using it in application to real systems [14].

T, one obtains the values of $c_{l_1,l_2,\ldots,l_D}$, $a_{l_1,l_2,\ldots,l_D}$, $b_{l_1,l_2,\ldots,l_D}$, and an approximation error ε by the linear least squares method. Then, the trial value of T is varied through a certain range near the initial estimate $T=T^*$ and approximation is performed for each of the trial values. The graph of $\varepsilon(T)$ has a sharp and deep minimum which corresponds quite precisely to the ''true'' value of the driving period.

Note, that rather high accuracy of determining the driving period value is needed. Otherwise, a significant ''phase difference'' between original driving and its approximation $a\cos[(2\pi/T)t]+b\sin[(2\pi/T)t]$ occurs during the investigated time interval $T_N=N\Delta t$. One can derive an estimate for an accessible error ΔT of determining the value of the driving period (from the condition that the relative error of driving approximation does not exceed ε_0):

$$\frac{\Delta T}{T}\leqslant\varepsilon_0\frac{\sqrt{3}}{2\pi}\frac{T}{T_N}.$$

The precise calculation of values of certain driving parameters is, in general, necessary when nonautonomous equations are reconstructed.

III. APPLICATION OF THE MODIFIED MODEL STRUCTURE

Further, we illustrate the advantages of the suggested approach by four numerical and one natural examples. All numerical time series were obtained by integrating numerically known systems of ODE's (''objects of modeling'') using the fourth-order Runge–Kutta routine with the step $\Delta t=0.01$. In each case, a time series contained 6000 values (about 10 driving periods). An original time series in the physical experiment was obtained by analogous-digital conversion of a realization of potential. A standard system (1) and (2) and a nonautonomous system (5) and (6) are reconstructed from the time series. Results achieved by using a model with additive driving (3) are also presented. For each model structure only the best (corresponding to the most effective model) results are reported.

To check *qualitative similarity* between an original and a model behavior, a model phase orbit is compared to a phase orbit reconstructed from a time series. The goodness of *quantitative description* is estimated as an interval of a sufficiently accurate short-term prediction according to [17]. First, the value of $\sigma(\tau)$—root-mean-squared error[2] of predicting a time interval τ ahead (normalized to the standard deviation of an observable v)—is calculated. Second, a time interval τ_{pred} at which the value of $\sigma(\tau)$ becomes more than a certain threshold value σ_c is found. We choose the threshold value $\sigma_c=0.05$. The time interval τ_{pred} is called *prediction time*. This value is always presented in units of the driving period because the latter is a characteristic time scale for all examples.

The first example illustrates the efficiency of the proposed modification in the presence of noise (in the case of additive driving). An observed time series is a chaotic time realization of the coordinate y_1 of harmonically driven Duffing oscillator:

$$\dot{y}_1=y_2,$$

$$\dot{y}_2=-\gamma_0 y_2-y_1-y_1^3+A_0\cos\omega_0 t, \tag{7}$$

where $\gamma_0=0.1$, $A_0=35$, $\omega_0=1$. This system of ODE's can be written in the form (1) and (2) with $D=4$ and $K=3$. To see this, it is sufficient to differentiate the second equation twice and replace the term $A_0\cos\omega_0 t$ by its expression from (7). Thereby, one derives

$$\dot{y}_1=y_2,$$

$$\dot{y}_2=y_3,$$

$$\dot{y}_3=y_4,$$

$$\dot{y}_4=-\gamma_0 y_4-(1+3y_1^2+\omega_0)y_3-6y_1y_2^2-\omega_0^2\gamma_0 y_2 - \omega_0^2(y_1+y_1^3). \tag{8}$$

The best model (1) and (2) is obtained, naturally, at $D=4$, $K=3$. The best model (5) and (6) is obtained at $D=2$, $K=3$. All model coefficients agree within a percent with the original coefficients in both cases. Prediction time is large enough for both models: $6T$ and $7T$, respectively.

The standard method is also effective because the structure of the original system (8) and the model structure (1) and (2) completely coincide. However, to build an effective standard model, one needs to calculate four time derivatives of an observable, while to construct a nonautonomous model, it is necessary to find only two derivatives. Therefore, when a noise (namely, Gaussian white noise with the standard deviation of about 0.001 of the signal standard deviation) is added to the same time series, the standard method is no longer effective (the values of model coefficients are far from original ones) due to huge errors of numerical differentiation, while a model (5) and (6) remains quite efficient (the values of model coefficients are close to the original ones). The prediction time is small for both models, but for a nonautonomous model it is 10 times greater than for a standard one: $0.5T$ and $0.05T$, respectively. Note, that a model (3) is the most relevant in the case considered since its structure is exactly the same as the form of the original system (7). So, a model (3) does not contain several superfluous coefficients which are incorporated in (5) and (6). However, the results which are achieved by using a model (3) almost do not differ from the above-mentioned results for the model (5) and (6).

The second example illustrates a situation when a model (5) and (6) is efficient while the standard approach does not lead to success because the dependence (1) is too complex to

[2]The average is carried over different initial conditions. These initial conditions are taken from the part of a time series which is not used for a model construction; they are always the same for an object and for a model.

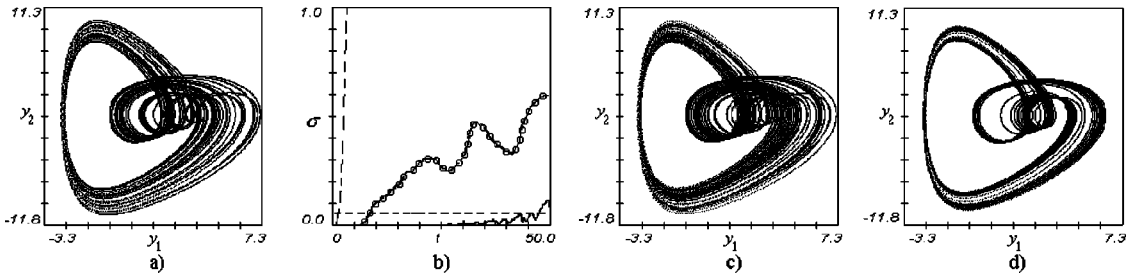

FIG. 1. (a) A projection of the attractor of Toda oscillator (10). (b) Prediction error for the model (1) and (2) with $D=4$ and $K=6$ (the dashed line), for the model (3) with $D=2$ and $K=10$ (the circles) and for the model (5) and (6) with $D=2$ and $K=9$ (the solid line). (c) and (d) Phase orbits of the model (5) and (6) and of the model (3), respectively. Phase orbits of the standard model are globally unstable.

be approximated by a polynomial. Models (3) are not effective also since taking only additive driving terms into account is not sufficient.

An observed time series is a chaotic time realization of the coordinate y_1 of harmonically driven Toda oscillator [a projection of the phase orbit is shown in Fig. 1(a)]:

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = -0.45y_2 + (5 + 4\cos t)(e^{-y_1} - 1) + 7\sin(t). \quad (9)$$

Similarly to the first example, it is possible to show that the equations (9) can be written in the form (1) with $D=4$ and a rather complicated function on the right-hand side:

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = y_3,$$

$$\dot{y}_3 = y_4,$$

$$\dot{y}_4 = -0.45y_4 + (y_3 + 0.45y_2)\left(\frac{8}{7}y_2 e^{-y_1} - 1\right) + e^{-y_1}\left(y_2^2 - y_3 - \frac{8}{7}y_2(e^{-y_1} - 1)\right)$$

$$\times \frac{\frac{16}{7}(e^{-y_1} - 1)(y_3 + 0.45y_2) + 4(y_4 + 0.45y_3 - 10y_2 e^{-y_1}) + 35}{7 - 4y_2 e^{-y_1} + \frac{16}{7}(e^{-y_1} - 1)^2}. \quad (10)$$

Prediction times for the optimal models (5) and (6), (3), and (1) and (2) are equal to $7T$, $1.5T$, and $0.15T$, respectively. For this example, the graphs σ versus t (which allow for determining a prediction time) are shown in Fig. 1(b). The best model (5) and (6) is obtained at $D=2$, $K=9$ [Fig. 1(c)]. The values of its coefficients are quite close to the values of the corresponding coefficients of the original system (to compare these values, one should replace the exponential function by its Taylor series expansion). The optimal model (3) with $D=2$ and $K=10$ works essentially worse [Fig. 1(d)].

The optimal for prediction standard model with $D=4$ and $K=6$ demonstrates globally unstable orbits. This is due to the fact that effective approximation of complex dependence (10) in a space of high dimension by a polynomial is very difficult. The model is too big, estimates of the values of its coefficients are not reliable. A model orbit may go out of the region of the observed motion (due to unavoidable approximation errors). Then, the polynomial is no longer connected with the original system. As a result, a model orbit goes to infinity. Note that this, unsuccessful for the standard method result, appears even in the absence of noise.

The third example: harmonic driving enters equations of an original system in a linear additive way. Nevertheless, a model (3) does not work well, but a model (5) and (6) is quite effective.

An observed time series is a chaotic time realization of the x coordinate of the Rossler system, the z coordinate being driven

$$\dot{x} = -y - z,$$

$$\dot{y} = x + ay, \quad (11)$$

$$\dot{z} = b - cz + xz + A\cos t,$$

where $a=0.398$, $b=2.0$, $c=4.0$, $A=1.0$. The system (11) can be written in the form

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = y_3,$$

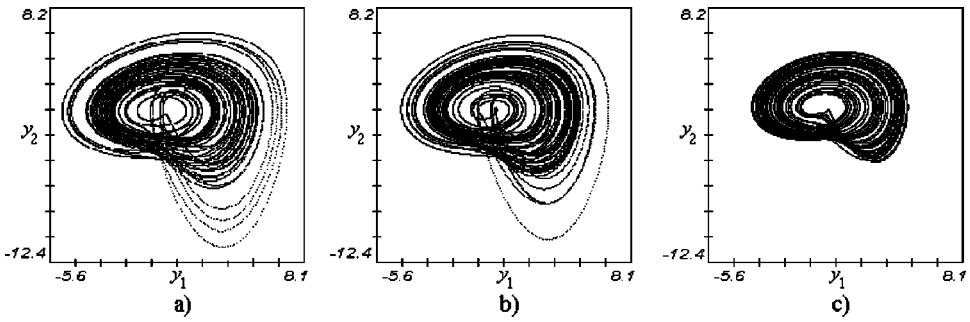

FIG. 2. (a) A projection of the attractor of Rossler system (12). (b) and (c) Phase orbits of the model (5) and (6) with $D=3$ and $K=5$ and of the model (1) and (2) with $D=5$ and $K=3$, respectively. The optimal model (3) with $D=3$ and $K=5$ has globally unstable orbits.

$$\dot{y}_3 = ab - cy_1 + (ac-1)y_2 + (a-c)y_3 + y_1^2 - ay_1y_2 + y_1y_3 - \frac{y_2(y_1+b-ay_2+y_3)}{a+c-y_1} + \frac{-(1-a)^2A\cos t + (1-c^2-2cy_1-y_2-y_1^2)A\sin t}{a+c-y_1}, \quad (12)$$

where $y_1=x$ [a projection of the phase orbit is shown in Fig. 2(a)]. Prediction times for the optimal models (5) and (6), (3), and (1) and (2) are equal to $4T$, $0.25T$, and $0.35T$, respectively. The results of comparison of different models showing the advantages of a model (5) and (6) are presented in Figs. 2(b) and 2(c) [phase orbits of the model (3) are globally unstable].

The fourth example shows that a model (5) and (6) can be effective also when higher powers of driving terms enter equations of an original system.

An observed time series is a chaotic time realization of the y coordinate of the nonautonomous Rossler system which differs from (11) only in that the y coordinate is driven instead of the z coordinate. Again, Rossler equations can be written in the form

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = y_3,$$

$$\dot{y}_3 = -b - aA^2/2 - cy_1 + (ac-1)y_2 + (a-c)y_3 - ay_1^2 + (a^2+1)y_1y_2 - ay_2^2 - ay_1y_3 + y_2y_3 + A((1-2a^2+ac)\cos t + (1+a-c-a^2)\sin t) - y_2A\sin t + y_3A\cos t + ((a+1)/2)A^2\sin(2t), \quad (13)$$

where $y_1=y$ [a projection of the phase orbit is shown in Fig. 3(a)]. Now, the function on the right-hand side of the last equation contains the term $\sin(2t)$ along with the first powers of driving terms. This term, however, does not affect significantly qualitative behavior of the system. Its quantitative effect is also not great. Therefore, our approach appears to be efficient. Prediction times for the optimal models (5) and (6), (3), and (1) and (2) are equal to $1T$, $0.25T$, and $0.3T$, respectively. A model (5) and (6) demonstrates an attractor which is similar to the original one [Fig. 3(b)]. The orbits of the standard model and of the model (3) are globally unstable. This example shows that the model structure (5) and (6) is useful for the description of a sufficiently wide class of harmonically driven systems (when mainly the first powers of driving terms affect the original dynamics).

Efficiency of this approach for modeling real systems is illustrated in Fig. 4 where the results of modeling a radiotechnical circuit—a circuit with switched capacitors [18,13]—are presented. This system can be described, in brief, as follows. The circuit is driven by an external harmonic signal with the amplitude U_0 and frequency ν [see the scheme in Fig. 4(a)]. The element K of the scheme is an electronic switch—a microscheme comprising dozens of transistors and other passive elements, which is fed from a special source of dc potential. When the value of potential U

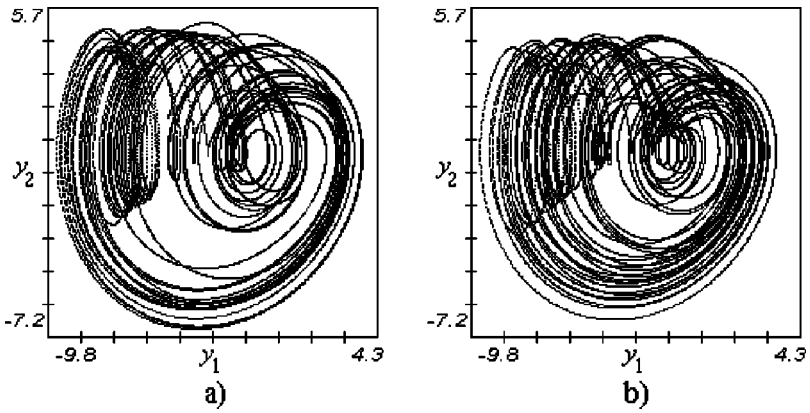

FIG. 3. (a) A projection of the attractor of Rossler system (13). (b) A projection of the attractor of the model (5) and (6) with $D=3$ and $K=5$. The model (1) and (2) with $D=4$ and $K=3$ and the model (3) with $D=3$ and $K=2$ demonstrate globally unstable orbits.

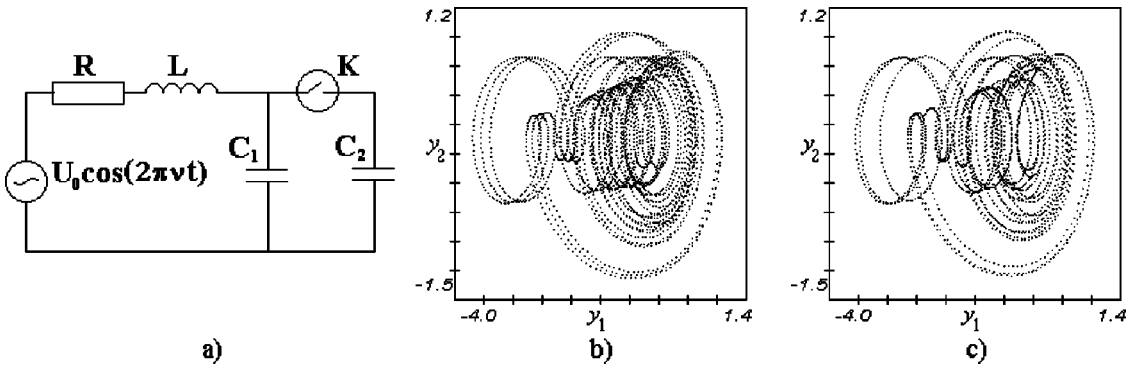

FIG. 4. (a) A circuit with switched capacitors (K is a switch). The values of the parameters of the circuit are following: $R=10\Omega$, $L=14$ mH, $C_1=0.1$ μF, $C_2=4.4$ μF, the value of the threshold potential $U_{\rm thr}=-0.2$ V, $U_0=2.3$ V, $\nu=3$ kHz, a sampling frequency is 250 kHz. (b) A projection of the attractor of the experimental system. The values y_2 and y_1 are proportional to the current through the resistor R and to its integral, respectively. They are presented in arbitrary units. (The time series observed contains numbers proportional to the values of potential on the resistor. They come from the output of an ADC in the range from -2048 to 2048 and are divided by 600.) (c) A projection of the attractor of the model (5) and (6) with $D=2$ and $K=5$.

on the capacity C_1 is small, linear oscillations in the circuit RLC_1 occur (the resistance of the switch is very large). When the value of U increases through a threshold value $U_{\rm thr}$, the resistance of the switch decreases abruptly and the capacity C_2 is connected to the circuit.

Models were constructed from an integrated chaotic time realization of the current through the resistor R [Fig. 4(b)]. Standard polynomial models cannot predict the measured series further than $1T$ ahead and demonstrate either globally unstable or simple periodic phase orbits. A model (5) and (6) provides the prediction time $6T$ [while the prediction time for a model (3) is less than $5T$] and its behavior is qualitatively similar to the experimental one [Fig. 4(c)].

It is possible, of course, to find examples of systems for modeling of which our approach is not useful. Such is a system (5) with a complicated dependency of the function f on the driving terms when it is not sufficient to take into account only their first powers. Another example is a system of very high dimension when too many equations would be needed for modeling. In the latter case, our method suffers from the same shortcomings as the standard approach does. Nevertheless, the suggested approach can, obviously, be useful for many practical situations.

IV. CONCLUSIONS

The success of modeling from a time series is determined by the choice of informative dynamical variables, a corresponding *structure of model equations* and a method of calculating their coefficients. Obviously, one can hope for useful recommendations only with respect to sufficiently small classes of objects to be modeled.

In this paper, a structure of equations for modeling harmonically driven systems is proposed. Our method is a modification of the standard global reconstruction technique. The form of model equations is selected by taking into account *a priori* (or obtained experimentally) information about the presence of harmonic driving. Namely, explicit time dependence is incorporated in model equations. Such an approach has certain advantages in comparison with the standard one. It allows for using fewer model equations and, hence, calculating fewer time derivatives of an observable and model coefficients. Efficiency of the approach is illustrated by several examples with both additive and parametric harmonic driving.

The modification of the standard method is considered in the paper for the case of modeling from *a scalar time series* and using *the method of sequential derivatives* to reconstruct state vectors. One can, however, employ it when the method of time delays is used. The only difference is that it would be necessary to fit D various functions. Hence, one should include driving terms into each of them according to (6). In the same manner, application of the modification is possible when several variables (i.e., a vector time series) are measured. Thus, the proposed method seems to be widely applicable.

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