Experimental observation of dynamics near the torus-doubling terminal critical point

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A nonlinear electronic circuit driven by two signals with golden mean frequency ratio is studied. We have observed different dynamical regimes, including tori, doubled tori, strange nonchaotic attractors, and chaos, which all meet together at the critical point terminating the torus-doubling bifurcation curve (TDT point) in the parameter plane of the system. The parameter plane arrangement, spectra, and portraits of attractors observed in the experiment are in reasonable coincidence with numerical computations for a quasiperiodically forced logistic map, where the TDT critical point originally has been found and studied theoretically [S. Kuznetsov, A. Pikovsky, and U. Feudel, Phys. Rev. E **57**, 1585 (1998)].

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Recently much attention of the researchers in nonlinear dynamics is directed to strange nonchaotic attractors (SNAs), which appear typically in nonlinear systems driven by external quasiperiodic force in intermediate region between order and chaos. These attractors have only nonpositive Lyapunov exponents, but their geometrical structure appears to be fractal due to high sensitivity of dynamics in respect to phase of the external force. Originally SNA's were described by Grebogi *et al.* in 1984 [1], and since then investigated extensively both theoretically (e.g., Refs. [2–7]), and experimentally (e.g., Refs. [8] and [9]).

A particularly interesting question is the onset of strange nonchaotic behavior and its connections with the onset of chaos [10–15]. In two articles of Kuznetsov, Pikovsky, and Feudel [11,15] this problem was analyzed by means of the renormalization-group approach, similar to that developed earlier in context of the onset of chaos [16–18]. In the model of the logistic map under external driving with the golden mean frequency [19–21]

$$x_{n+1}=\lambda-x_n^2+\epsilon\sin(2\pi n\nu),\qquad \nu=(\sqrt{(5)}-1)/2 \tag{1}$$

the critical point was found, where domains of torus, doubled torus, SNA, and chaos meet together in the parameter plane. It is located at the terminal point of the bifurcation curve of torus doubling [19,22,23]. At this TDT point, instead of the smooth torus the fractal object (the critical torus) is present. The neighborhood of the parameter plane near the TDT point possesses a property of self-similarity, or scaling [15]. Namely, using appropriately choosen local coordinates, one can observe reproducing the structure of the parameter plane infinitely often in smaller and smaller scales near the critical point.

Due to the renormalization-group arguments presented in Ref. [15], it may be conjectured that the observed type of behavior can be found in many other systems relating to the same universality class as the forced logistic map. Obviously, it would be interesting to observe the TDT critical behavior in experiments with systems of different physical nature: Any nonlinear dissipative system is appropriate, if it demonstrates the Feigenbaum's period-doubling cascade, and the quasiperiodic external force is added.

Since the work of Linsay [24], a well-known example of nonlinear dissipative dynamics is the nonlinear electronic circuit, containing an inductance, resistor, and a *p*-*n* junction (semiconductor diode) [25,26]. Being excited by alternating voltage of appropriate frequency ω_1 this circuit manifests the period-doubling cascade, as the amplitude of driving A_1 is increased. According to Feigenbaum's concept of quantitative universality [16], logistic map $x_{n+1}=\lambda-x_n^2$ may serve as an appropriate phenomenological model for this system.

Let us add one more component of the external excitation, which has amplitude A_2 and frequency ω_2, setting $\omega_2/\omega_1=[\sqrt{(5)}-1]/2$. Now the phenomenological model will be, apparently, Eq. (1). The value of A_1 will play a role of the control parameter λ, while A_2 will correspond to the amplitude of driving ϵ.

Figure 1 presents the schema of our experiment. It contains the (resistor inductor) RL-diode circuit, two generators

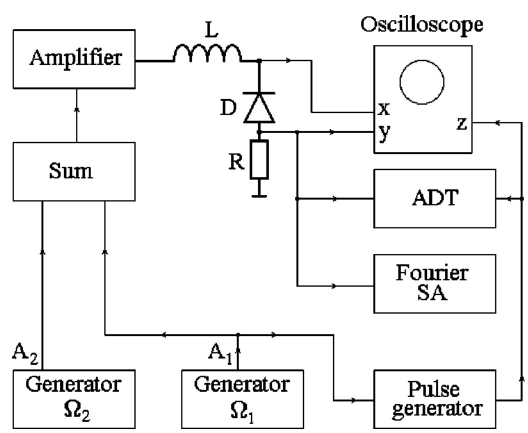

FIG. 1. Scheme of the experiment with the nonlinear electronic circuit under quasiperiodic external driving. Two generators produce alternating voltage of amplitudes A_1 and A_2, and frequencies ω_1 and ω_2. Ratio of frequencies normally is equal to the golden mean, although in special cases may be ensured to be a rational approximant to it. Values of A_1 and A_2 are regarded as main control parameters of the system.

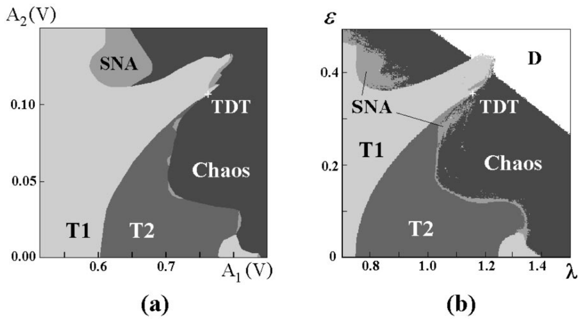

FIG. 2. Charts of dynamical regimes obtained in the experiment (a) and computed for the forced logistic map (b). T1, T2, and T4 are domains of existence of smooth tori, arisen one from another, via the torus-doubling bifurcations, SNA is strange nonchaotic attractor, C is chaos, and D is domain of divergency of iterations for the logistic map. The TDT critical point is marked by a little cross. For the experimental system amplitudes of two components of the external force, A_1 and A_2 are plotted along the coordinate axes, for the logistic map coordinates are the control parameter λ and the amplitude of the external force ϵ.

of sinusoidal alternating voltage, summator, and an amplifier with low-output impedance. Frequency ratio was controlled with the precision of four decimal digits.

We used several approaches to distinguish types of dynamics observed at different values of parameters in the experiment.

The Fourier spectrum of the output signal could be observed on a screen of the spectrum analyzer, which allowed, in particular, visual identification of the chaotic regimes (continuous spectrum).

Observation of phase portraits on the screen of the oscilloscope gave a possibility to find bifurcations of torus doubling.

Stroboscopic Poincaré section portrait of attractor could be observed using short-time pulse control of the oscilloscope beam, so that the images of points appeared on the screen following with time interval of $2\pi/\omega_1$. Then, the torus is represented by a smooth closed curve, and the doubled torus—by two smooth closed curves, and so forth. Loss of smoothness of the observed curves, or their smearing, could be associated with the onset of SNA or chaos.

To distinguish more carefully regimes of the smooth torus and of SNA we adopted an experimental version of criterion suggested in Ref. [6]. Namely, using operations of multiplicating and division of the basic frequency ω_1, we could ensure frequency ratio equal to a rational approximant of the golden mean, namely, 13/21, and control presence or absence of bifurcations in the system in dependence on the relative phase between two components of the external driving. Presence of the bifurcations was regarded as indication of the SNA.

In Fig. 2 two charts of dynamical regimes are shown, one obtained directly from the experiment and another computed for the forced logistic map (1), as in Ref. [15]. For the first case, the diagram is drawn on the plane of two amplitudes of the external signal (A_1,A_2), corresponding to the frequencies ω_1 and ω_2; for the second case—on the plane (λ,ϵ). Domains of distinct regimes are shown by gray tones, and the nature of each regime is indicated in the diagrams by letters.

By the analog-digital transformer, the output signal of the system, proportional to voltage on the resistor, could be introduced into a computer and processed. In particular, using data for output voltage with time step equal to one period of the basic frequency $2\pi/\omega_1$, we plotted them in coordinates (U_n,U_{n+1}) to obtain the iteration diagrams. In Fig. 3(a) some examples of such diagrams are presented, corresponding to regimes of torus, doubled torus, SNA, and chaos. They may be compared with Fig. 3(b), where similar plots for the logistic map are shown.

Both charts in Fig. 2 look remarkably similar. In particular, one can see there the torus-doubling bifurcation

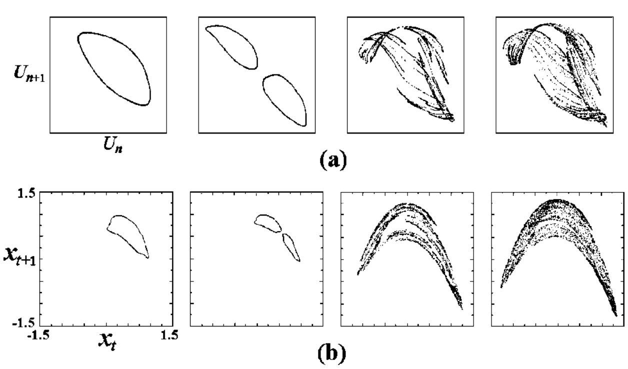

FIG. 3. Examples of iteration diagrams obtained by computer processing of data from the experimental system (a) and from computations for the logistic map (b). Diagrams I, II, III, and IV relate to torus, double torus, SNA, and chaos, respectively.

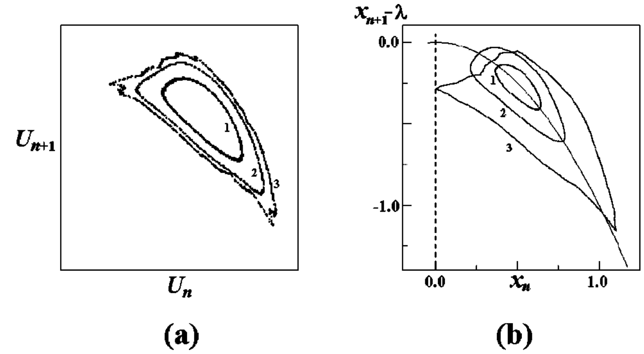

FIG. 4. Evolution of the iteration diagram while moving along the torus-doubling bifurcation curve towards the TDT critical point: (a) experiment and (b) computation for the logistic map. Observe fractal-like form of the curve representing the critical attractor.

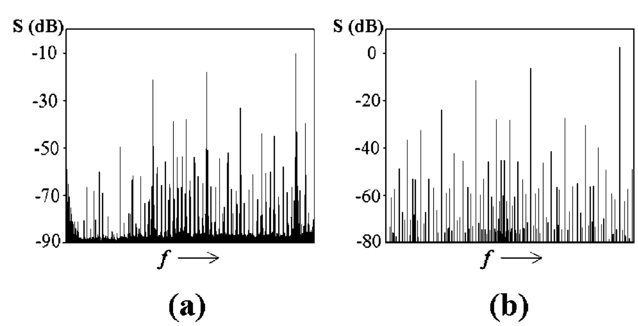

FIG. 5. Fourier spectrum obtained exactly at the TDT critical point from experiment (a) and numerically from the logistic map (b).

line, which separates regions T1 (torus) and T2 (doubled torus). In experiment, tuning simultaneously two parameters A_2 and A_1, it was sufficiently easy to move along the bifurcation line and find its terminal point, i.e., the TDT critical point. In Fig. 4(a) we demonstrate how the iteration diagram of the dynamics evolves in this process. An analogous plot for the logistic map is reproduced in Fig. 4(b) [15]. Observe that at the TDT point the attractor accepts a very specific form, similar both in the experimental system and in the logistic map: It has a fractal-like shape with a break at the leftmost point. Fourier spectra, exactly at the critical point for the experimental system and for the logistic map, are also in reasonable qualitative correspondence (Fig. 5).

To conclude, in the experiment we have confirmed results of the theoretical analysis concerning existence of the TDT critical point, parameter plane arrangement near this point, and associated peculiarities of the dynamics. Apparently, our experimental system indeed relates to the same universality class as the quasiperiodically forced logistic map. It may be conjectured that dynamics of many other realistic systems of a different nature under quasiperiodic driving can manifest analogous behavior.

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