New type of critical behavior of coupled systems at the transition to chaos

B. P. Bezruchko, Academician Yu. V. Gulyaev, S. P. Kuznetsov, and E. P. Seleznev

Saratov Branch of the Institute of Radiotechnology and Electronics, Academy of Sciences of the USSR; N. G. Chernyshevskii State University, Saratov

(Submitted June 22, 1985)

Dokl. Akad. Nauk SSSR **287**, 619–622 (March 1986)

1. It has been established that simple nonlinear oscillatory systems can demonstrate chaotic dynamics in the absence of any kind of external random perturbation. Significant progress has recently been made in understanding this phenomenon. This progress is based on the study of the laws of transition from regular behavior to chaos or to a variation of the parameters of the system.[1–3] The critical phenomena (i.e., the phenomena near the transition point) are associated with large time scales, which greatly exceed all the other characteristics times of the system. As a result, the critical phenomena have the properties of universality and similarity, which are determined only by the qualitative class of the transition, and not by the detailed form of the equations of the system, which is associated with the local-time features of the dynamics.

The most thoroughly studied type of critical behavior corresponds to the order–chaos transition that takes place by a sequence of temporal period-doubling bifurcations of a motion.[1] The simplest example of such a system is the one-dimensional recurrence mapping

$$x_{n+1} = 1 - \lambda x_n^2, \tag{1}$$

where x_n is the variable that defines the state of the system at the discrete time n, and λ is a parameter. The critical value of the parameter for the system (1) is $\lambda_c = 1.40116$. The main property of similarity of critical phenomena consists in the following: when λ approaches the critical point, $\delta = 4.6692$ times a regime is realized in the system that is similar to the initial one, but with a doubled time scale (both at $\lambda < \lambda_c$ and at $\lambda > \lambda_c$). In other words, the partitioning of the λ axis into regions of different regimes has the property of scale invariance, and transforms into itself when the scale is changed with respect to the point λ_c by a factor of δ. Diverse systems that are described by differential equations, such as the nonlinear dissipative oscillator excited by an external period force,[6] at the transition to chaos demonstrate an analogous behavior, which is described by the same universal scaling constant δ. As a result of the universality of critical phenomena, it is always possible to use model (1) to describe any system near the transition point to chaos by period doublings, if the appropriate meaning is given to the variable x_n and the parameter λ.[1]

In this study we report the discovery of a new type of critical behavior in a system consisting of two subsystems (each of which demonstrates period doubling) that have a one-way coupling between them (the first subsystem acts on the second subsystem, but the second subsystem does not

 0038-5689/86/03 0258-03 $02.20

influence the first subsystem). Recently, similar models have attracted attention in connection with the investigation of turbulence that develops downstream.[5]

2. We consider the following model system for coupling mappings:

$$x_{n+1} = 1 - \lambda_1 x_n^2, \quad y_{n+1} = 1 - \lambda_2 y_n^2 - \beta\varphi(x_n), \tag{2}$$

where x_n and y_n are the variables that characterize the states of the first and second subsystems, respectively; λ_1 and λ_2 are the controlling parameters of the subsystems; and β is the coupling parameter. The function $\varphi(x)$ characterizes the way in which the coupling is introduced, and in the simplest case can be chosen to be x or x^2.

Figure 1 shws the numerically obtained region configuration in the (λ_1, λ_2) parameter plane for the case $\phi(x) = x$ and $\beta = 1/4$. As the parameter λ_1 is increased, the first subsystem exhibits a Feigenbaum sequence of period-doubling birfurcations of the stable cycles. The corresponding bifurcation lines are shown in Fig. 1 by vertical lines. For small values of the parameter λ_2, the period of motion of the second subystem is the same as that of the first subsystem (forced oscillations). If λ_2 is increased at fixed λ_1, a sequence of period-doubling bifurcations will be observed in the second subsystem. The bifurcation values of λ_2 depend on λ_1 (the curves in Fig. 1). The numbers inside the different regions indicate the period of the oscillations in the second subsystem, and the hatching indicates the onset of the chaos.

Figure 1 shows that the parameter space of the system has a scale-invariant structure: the whole pattern of regions which is shown is reproduced on a smaller scale inside the rectangle formed by the dashed lines. The corresponding change of scale along the λ_1 axis is proportional to the Feigenbaum constant $\delta_1 = 4.6692$, and the change of scale along the λ_2 axis is determined by a new constant, which we numerically find to be $\delta_2 = 2.39$.

We call the symmetry center of the pattern of the regions in the (λ_1, λ_2) plane the bicritical point. At this point the system has a countably infinite set of (unstable) cycles of period 2^N. The elements of the cycles that approach zero in the first subsystem vary in proportion to α_1^{-N}, where $\alpha_1 = 2.5029$, and in the second subsystem they are proportional to α_2^{-N}, where $\alpha_2 = -1.52$. The multipliers of the 2^N cycles are the same at large N: a small perturbation of the variable x varies by $\mu_1 = -1.6012$ times in a period of the cycle, and that of the variable y varies by $\mu_2 = -1.176$ times.

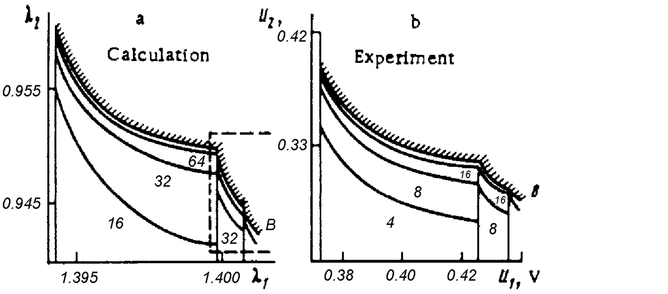

FIG. 1. Parameter plane for (a) mapping (2) and for (b) coupled nonlinear oscillatory contours. B is bicritical point.

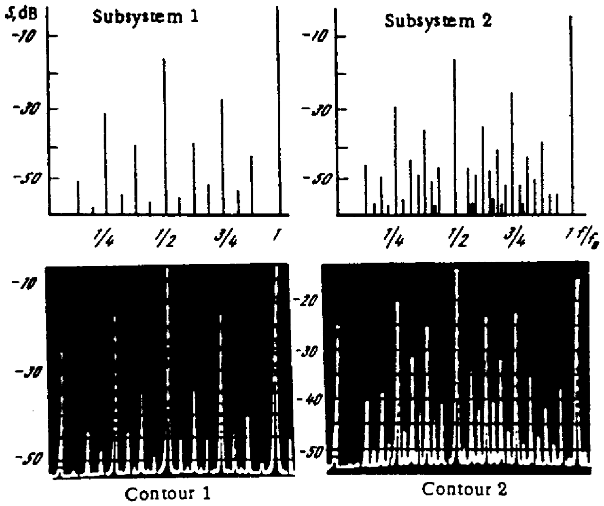

FIG. 2. Oscillation spectra in the two subsystems at the critical point: Above – calculation for mapping (2); below – experiment on coupled nonlinear oscillatory contours.

The attractor of the system at the bicritical point corresponds to the motion with an infinite period. Figure 2 shows the corresponding oscillation spectra of the subsystems. The first subsystem exhibits a classical Feigenbaum spectrum with a gradient of about $\gamma_1 = 13.6$ db between the spectral components of each successive level. In the spectrum of the second subsystem, the ratio between the levels of the subharmonics is completely different and is characterized by the constant $\gamma_2 \sim 6$ db.

We note that the position of the bicritical point on the line $\lambda_1 = 1.40116$ depends on the magnitude of the coupling (the parameter β) and on the form of the smooth function $\varphi(x)$, but the scaling constants introduced above do not depend on them. This circumstance allows us to assume that δ_2, α_2, μ_2, and γ_2 are universal just as the Feigenbaum constants δ_1, α_1, μ_1 and γ_1. Consequently, these constants should appear in all cases in which a period-doubling system affects another such system, irrespective of whether these systems are described by mappings or differential equations. In order to check this proposition, we carried out an experiment on a specific system of coupled nonlinear oscillators (oscillatory contours) excited by an external periodic force.

3. The system under investigation consisted of two identical nonlinear capacitances of the p–n junctions of two KP903 transistors. The coupling between the contours was realized by using a special amplifier such that the first contour influenced the second contour, but the second contour did not influence the first contour. Each contour was excited by in-phase sinusoidal signals (from an external oscillator) whose amplitudes could be regulated independently.

As the amplitude U of the external force was increased each contour exhibited, in the absence of a coupling, a sequence of period-doubling bifurcations, which culminated in a transition to chaos at a certain critical value $U = U_c$. In the region $U < U_c$ it was possible to clearly identify five bifurcation doublings, which correspond to the appearance of oscillations with a period of $32T_0$ (T_0 is the period of the external force). As U was increased in the region $U > U_c$, the discrete spectral lines disappeared gradually (with regions of continuous spectra forming in their place) in the order opposite to that in which they appeared. We also observed a "window of stability", i.e., a band in the region $U > U_c$ in which periodic oscillatory regimes were realized with periods of $3 \cdot 2^N T_0$, $5 \cdot 2^N T_0$, etc. The evolution of the oscillation spectrum as a function of U, and also the estimates of the scalling constants, correspond to the results of Feigenbaum's theory.

The introduction of a coupling did not produce any changes in the oscillations of the first contour. The dynamics of the second contour was determined by the parameters U_1 and U_2, i.e., the amplitudes of the external force on each of the two contours, and also by the magnitude of the coupling between the contours. On the right side in Fig. 1 we show the experimentally determined regions of the various oscillatory regimes in the (U_1, U_2) parameter plane at a fixed value of the coupling. The oscillation periods are expressed in units of T_0. The hatching indicates the boundary at which chaos starts. The region pattern obtained in the experiment is in good qualitative agreement with that determined for the model mapping (2), although the expected properties of scale invariance hold only approximately. This circumstance clearly stems from the fact that in the experiment the resolving power of the apparatus allows us to fix a relatively small number N of period doublings, whereas the universal properties of similarity are valid asymptotically in N.

By varying the two parameters U_1 and U_2 in the experiment, it was easy to find the bicritical point, whose characteristic property is the transition to chaos in the first contour for an arbitrarily small increase in U_1, and a transition to chaos in the second contour for an arbitrarily small increase in U_2. Here we show the photographs of the spectra of the oscillations of each contour at the bifurcation point (Fig. 2). These spectra exhibit a remarkable agreement with the calculated spectra for the model mapping (2). By varying the coupling of the contours within wide limits, it was possible to observe the bicritical point, together with its inherent properties of the signal spectrum, and to observe the characteristic structure of the neighboring regions of the parameter space.

The results of this study show graphically that the universality of the laws of critical behavior are expressed not only in the dynamics of an individual system that exhibits a transition to chaos by period doubling, but also in phenomena arising from the interaction of such systems.

[1]M. Feigenbaum, Usp. Fiz. Nauk 141, 343 (1983).
[2]Y. Romeau and P. Manneville, Comm. Math. Phys. 74, 189 (1980).
[3]D. Rand, S. Ostlund, J. Sethna, and E. Siggia, Phys. Rev. Lett. 49, 132 (1982).
[4]A. N. Sharkovskii, Ukr. Mat. Zh. 16, 61 (1964).
[5]A. V. Gaponov-Grekhov, M. I. Rabinovich, and I. I. Starobinets, Pis'ma Zh. Eksp. Teor. Fiz. 39, 561 (1984) [JETP Lett. 39, 688 (1984)].
[6]J. Testa, J. Perez, and C. Jaffries, Phys. Rev. Lett. 48, 714 (1982).

Translated by S. Monaghan

 0038-5689/86/03 0260-04 $02.20