Transfer entropies within dynamical effects framework

Dmitry A. Smirnov [*]
Saratov Branch, Kotelnikov Institute of Radioengineering and Electronics of Russian Academy of Sciences, 38 Zelyonaya St., Saratov 410019, Russia

(Received 18 April 2020; revised 5 October 2020; accepted 1 December 2020; published 23 December 2020)

Transfer entropy (TE) is widely used in time-series analysis to detect causal couplings between temporally evolving objects. As a coupling strength quantifier, the TE alone often seems insufficient, raising the question of its further interpretations. Here the TE is related to dynamical causal effects (DCEs) which quantify long-term responses of a coupling recipient to variations in a coupling source or in a coupling itself: Detailed relationships are established for a paradigmatic stochastic dynamical system of bidirectionally coupled linear overdamped oscillators, their practical applications and possible extensions are discussed. It is shown that two widely used versions of the TE (original and infinite-history) can become qualitatively distinct, diverging to different long-term DCEs.

DOI: 10.1103/PhysRevE.102.062139

I. INTRODUCTION

Transfer entropy (TE) is a celebrated concept which has become extremely popular in time-series analysis [1,2] during the two decades since its invention [3,4]. It is described as "directed statistical coherence" [5] between signals $\mathbf{x}_t$ and $\mathbf{y}_t$ or, in other terms, as "information transfer" [6,7] or "information flow" [2] between systems X and Y. TE has many extensions [8–16] and appears so influential to a large extent due to applications to causal coupling estimation from time series, which arises everywhere [2] from nuclear reactors [17] to neuroscience (e.g., a book [7]) and climate science (e.g., a review [18]).

Theoretically, a stochastic dynamical system Y with observed state vector $\mathbf{y}_t$ influences another system X with observed state vector $\mathbf{x}_t$, if and only if the TE $T_{Y\to X} > 0$, e.g., Ref. [2]. This circumstance makes $T_{Y\to X}$ relevant to *detect* an influence (causal coupling) $Y \to X$. In practice, one detects an influence $Y \to X$, if an estimate of $T_{Y\to X}$ is statistically significantly greater than zero, see, e.g., Refs. [19–32], for estimation techniques. This inverse problem is not easy, and cautions must be taken in respect of distinction between anticipation and causation [33–35], spurious causalities due to incompleteness of observations [36], interpretational difficulties due to synergy [10,37–41], etc. Still, one often tries to go even further and use obtained numerical values of $T_{Y\to X}$ measured in "nats" or "bits" to *quantify* the coupling $Y \to X$. It may readily lead to controversies since such quantifier of causal coupling does not always agree with "intuitive notion of causality." The latter term has been used in a discussion of spectral causalities [42] and seemingly implies some "variation-response" relationships, when a variation in a parameter of coupling $Y \to X$ or in an initial state or a parameter of the system Y induces a change in observed characteristics of X. Such relationships are formalized as "dynamical causal effects" (DCEs) for stochastic dynamical systems [43–46], in agreement with Pearl's interventional viewpoint [33,34,47].

To avoid the above controversies, one often explicitly claims that TE should not in general be interpreted as a causality measure but only as information transfer or flow, e.g., Ref. [2]. Also, one says that TE does not measure "causal mechanism" but only "causal effect" [48], which is understood as the effect of taking the data from Y into account when predicting the future of X. However, this is an effect of a researcher activity, not of a change in a system under study. It may be called an "informational" causal effect in contrast to "dynamical" causal effects. If TE were related to DCEs under some conditions, then one might use that in practice to interpret numerics of TE in a richer way and evaluate unknown DCEs from available estimates of TE. Relating TE to relevant DCEs is the purpose of this work, similarly to consideration of spectral causalities within the DCEs framework [46].

From its formal side, this study is similar to relating TE in previous theoretical works to such quantities as Wiener-Granger (WG) causality [49], log-likelihood ratio [50], thermodynamical potentials [51], limits of computation [5], and Lempel-Ziv complexity [52] (not to DCEs). Concerning numerics, several works have compared various coupling quantifiers in respect of their sensitivity (e.g., Refs. [53–56]) and studied their dependence on control parameters of systems under investigation (e.g., Refs. [4,57–59]), including such analysis of TE in different regimes in order to identify dynamical transitions or spatial structures as discussed in chapter 5 of Ref. [2]. However, those works are not intended to any interpretations of concrete numerical values of TE.

The paper is organized as follows. Section II recapitulates the definition of TE and appropriate DCEs. Section III describes "material and method" of the study: The former is a stochastic dynamical system of two bidirectionally coupled one-dimensional Langevin equations (overdamped oscillators), the latter is mostly analytic derivations in the form of

[*]smirnovda@yandex.ru

intermediate asymptotic as in the theory of dimension and similarity [60]. Section IV presents the results as characteristic relationships between the coupling quantifiers. Section V discusses their practical applicability and possibilities of generalization. Conclusions are given in Sec. VI. Technical details are in the Appendices.

II. COUPLING QUANTIFIERS

A. Transfer entropies

Consider a stationary Markov stochastic process $(\mathbf{X}_t, \mathbf{Y}_t)$: A vector $(\mathbf{x}_t, \mathbf{y}_t)$ is a complete state of the whole system (X, Y) in Markov sense [61,62], i.e., a vector $(\mathbf{x}_{t+\tau}, \mathbf{y}_{t+\tau})$ for any $\tau > 0$ does not depend on a more distant past, given $(\mathbf{x}_t, \mathbf{y}_t)$. Time t can be either continuous or discrete. Since empirical data typically take the form of a time series, the TE has been originally defined [3] for a discrete-time process $(\mathbf{X}_t, \mathbf{Y}_t)$ with $t = n\Delta t$, Δt is sampling interval and n is integer. Denote $p_{X,\tau|X}(\mathbf{x}_\tau|\mathbf{x}_0)$ conditional probability density function (pdf) of a future $\mathbf{x}_\tau$ given an initial state $\mathbf{x}_0$ of the subsystem X, and $p_{X,\tau|X,Y}(\mathbf{x}_\tau|\mathbf{x}_0, \mathbf{y}_0)$ its pdf conditioned by the complete initial state of the whole system (X, Y). Specify temporal horizon $\tau = \Delta t$. Denote (conditional) Shannon entropy of the pdf $p_{X,\tau|X}$ as $I_{X,\tau|X}$ and that of the pdf $p_{X,\tau|X,Y}$ as $I_{X,\tau|X,Y}$ (Appendix A). The former entropy quantifies uncertainty of "self-prediction," while the latter gives that of "joint prediction." The difference of the two uncertainties is the TE in the direction $Y \to X$:

$$T_{Y\to X,\tau} = I_{X,\tau|X} - I_{X,\tau|X,Y}. \tag{1}$$

It quantifies reduction of uncertainty in $\mathbf{x}_\tau$ if $\mathbf{y}_0$ is taken into account, given $\mathbf{x}_0$. TE equals zero if and only if the present of Y and the future of X are conditionally independent. TE is measured in nats if natural logarithm is used in the definition of Shannon entropy.

In practice, one is not informed *a priori* whether observables $\mathbf{x}_t$ and $\mathbf{y}_t$ constitute a complete state of the system. Empirical tests may reveal that they do not. So one forms (k, l)-histories $\mathbf{X}_t^{(k)} = (\mathbf{X}_t, \ldots, \mathbf{X}_{t-(k-1)\Delta t})$ and $\mathbf{Y}_t^{(l)} = (\mathbf{Y}_t, \ldots, \mathbf{Y}_{t-(l-1)\Delta t})$ [2] as proxies for state vectors and defines (k, l)-history TE (Eq. (4.11) in Ref. [2])

$$T^{(k,l)}_{Y\to X,\tau} = I^{(k)}_{X,\tau|X} - I^{(k,l)}_{X,\tau|X,Y}. \tag{2}$$

Here $I^{(k)}_{X,\tau|X}$ is (conditional) Shannon entropy of $p^{(k)}_{X,\tau|X}$ and $I^{(k,l)}_{X,\tau|X,Y}$—of $p^{(k,l)}_{X,\tau|X,Y}$, where $p^{(k)}_{X,\tau|X}[\mathbf{x}_{t+\tau}|\mathbf{x}_t^{(k)}]$ is conditional pdf of $\mathbf{x}_{t+\tau}$ given the history of $\mathbf{x}$ and $p^{(k,l)}_{X,\tau|X,Y}[\mathbf{x}_{t+\tau}|\mathbf{x}_t^{(k)}, \mathbf{y}_t^{(l)}]$ is the pdf conditioned by both histories. To use (k, l)-histories is very natural if one observes scalars x and y, as mentioned in Ref. [3].

In practice, one should find the pair (k, l) empirically, often from a short time series. Then several pairs (k, l) may seem to provide the generalized (k, l)-order Markov property and approximately the same uncertainty in the future $\mathbf{x}_{t+\tau}$. Different practical principles of the order selection may be used as discussed in Sec. 4.2.1 of Ref. [2], e.g., one can try various pairs (k, l) to select an optimum according to some criterion (say, minimal prediction uncertainty) or adaptively select only k using either $l = k$ or $l = 1$ [3]. Alternatively, one can first determine as large k as possible to use all information contained in the past of X and then select l. In theory, this principle implies $k = \infty$ and in general $l = \infty$. To use infinite history is exactly the idea of WG causality [63,64], where variances are used instead of Shannon entropies. However, in the most general formulation, Granger does not restrict this approach to variances [65], and so $T^{(\infty,\infty)}_{Y\to X,\tau}$ is a measure of WG causality [2,4]. For Gaussian processes, both characteristics are equivalent [49] as expressed via the formula $T^{(k,l)}_{Y\to X,\tau} = \frac{1}{2}\ln[1 + G^{(k,l)}_{Y\to X,\tau}]$, where $G^{(k,l)}_{Y\to X,\tau}$ is relative prediction improvement (Appendix A).

Returning to a first-order Markov process, the original TE $T_{Y\to X,\tau} = T^{(1,1)}_{Y\to X,\tau}$ and the infinite-history TE $T^{\infty}_{Y\to X,\tau} = T^{(\infty,\infty)}_{Y\to X,\tau}$ are two opposite cases in the full set of (k, l)-history TEs, as both of them adopt a simple qualitative formulation. For such process, it holds $T^{(k,l)}_{Y\to X,\tau} \geqslant T^{(k',l)}_{Y\to X,\tau}$ for any $k < k'$ and $l \geqslant 1$: Taking into account more distant past of X can improve its self-prediction but does not change uncertainty of joint prediction due to Markov property. Further, $T^{(k,l)}_{Y\to X,\tau} = T^{(k,l')}_{Y\to X,\tau}$ for any $k \geqslant 1$, $l \geqslant 1$, and $l' \geqslant 1$. Thus, $T_{Y\to X,\tau} \geqslant T^{\infty}_{Y\to X,\tau}$, i.e., the infinite-history TE is a lower bound for the original TE. Both TEs are often close to each other, e.g., Ref. [44].

B. Dynamical causal effects

Consider a stochastic dynamical system understood as a Markovian random dynamical system [66], i.e., $(\mathbf{X}_t, \mathbf{Y}_t)$ is a Markov process for any initial state $(\mathbf{x}_0, \mathbf{y}_0)$. In the time-series literature, it is also called a "state space model" [67]. Long-term DCEs are defined [43] for a parameterized system and quantify changes in the dynamics of X occurring after the coupling $Y \to X$ is switched on/off or an individual parameter of Y is changed. Such quantities are often of the main interest in practice, e.g., coupling effects on stationary variance in climate data analysis [44,68] or on power spectral density in neuroscience [42,46,69]. A system is specified with conditional (i.e., transition) pdf $p_{X,Y,\tau|X,Y}(\mathbf{x}_\tau, \mathbf{y}_\tau|\mathbf{x}_0, \mathbf{y}_0, \mathbf{a}_x, \mathbf{a}_{xy}, \mathbf{a}_y, \mathbf{a}_{yx})$, where parameters $\mathbf{a}_x$ and $\mathbf{a}_y$ relate to internal dynamics of the subsystems X and Y (e.g., relaxation rates, intrinsic noise intensities), while $\mathbf{a}_{xy}$ and $\mathbf{a}_{yx}$ specify couplings $Y \to X$ and $X \to Y$ (e.g., coupling coefficients). A widespread example serving as a paradigmatic system in theoretical studies (e.g., physical models of stochastic energetics [70]) is given by stochastic differential equations,

$$\begin{aligned}\dot{x} &= f_x(x, y, \mathbf{a}_x, \mathbf{a}_{xy}) + g_x(x, y, \mathbf{a}_x, \mathbf{a}_{xy})\xi_x(t),\\ \dot{y} &= f_y(y, x, \mathbf{a}_y, \mathbf{a}_{yx}) + g_y(y, x, \mathbf{a}_y, \mathbf{a}_{yx})\xi_y(t),\end{aligned} \tag{3}$$

where x and y are state variables, f_x and $f_y > 0$ are drift coefficients, g_x and g_y are diffusion coefficients, and (ξ_x, ξ_y) is Gaussian white noise with covariance intensity matrix $\mathbf{\Gamma}$ whose elements are constant Γ_{xx} and Γ_{yy} (Γ_{xx} is an element of $\mathbf{a}_x$ and Γ_{yy} of $\mathbf{a}_y$) with $\Gamma_{xy} = 0$.

The subsystem Y does not influence X, i.e., $p_{X,\tau|X,Y}$ does not depend on $\mathbf{y}_0$ for any $\tau > 0$, if and only if $\mathbf{a}_{xy} = 0$. Moreover, if $\mathbf{a}_{xy} = 0$, then $p_{X,\tau|X,Y}$ does not depend on $\mathbf{a}_y$. Stationary characteristics of $\mathbf{X}_t$ generally differ between $\mathbf{a}_{xy} = \mathbf{a}^*_{xy} \neq 0$ and $\mathbf{a}_{xy} = 0$. They may also differ between different values of $\mathbf{a}_y$ if $\mathbf{a}_{xy} \neq 0$. An effect of switching the coupling $Y \to X$ on, i.e., of changing $\mathbf{a}_{xy}$ from 0 to $\mathbf{a}^*_{xy} \neq 0$, is quantified as a respective change in a certain characteristic $Q\{\mathbf{X}_t|\mathbf{a}_{xx}, \mathbf{a}_{xy}, \mathbf{a}_{yy}, \mathbf{a}_{yx}\}$ which is a functional of stationary pdf of the process $\mathbf{X}_t$ [44,45] and, hence, a function of the system parameters. Omitting unchanged parameters, one defines

$$S_{Y\to X} = \frac{Q\{\mathbf{X}_t|\mathbf{a}_{xy} = \mathbf{a}^*_{xy}\} - Q\{\mathbf{X}_t|\mathbf{a}_{xy} = 0\}}{Q\{\mathbf{X}_t|\mathbf{a}_{xy} = 0\}}. \quad (4)$$

This quantity is called "coupling-on" long-term DCE.

For a system (3), effects of switching the noise ξ_y on are often of interest [43], especially in spectral causality studies [42,46]. A "noise-on" long-term DCE is given by

$$N_{Y\to X} = \frac{Q\{\mathbf{X}_t|\Gamma_{yy} = \Gamma^*_{yy}\} - Q\{\mathbf{X}_t|\Gamma_{yy} = 0\}}{Q\{\mathbf{X}_t|\Gamma_{yy} = 0\}}. \quad (5)$$

Both long-term DCEs are dimensionless and measured in relative units (r.u.) showing relative change of $Q\{\mathbf{X}_t\}$ under parameter variation. The value of 1 r.u. means doubling of Q. For zero-mean one-dimensional Gaussian processes, Q can be just stationary variance $Q\{X_t\} = \text{var}\{X_t\}$, which is then the simplest informative characteristic of the dynamics.

Both long-term DCEs are compared below to the two kinds of TE after reducing each TE to a single quantifier. Indeed, the original TE $T_{Y\to X,\tau}$ depends on τ and is, therefore, a family of coupling quantifiers, not a single one. For a typical case of nondegenerate noise intensity matrix $\mathbf{\Gamma}$, the TE $T_{Y\to X,\tau}$ is readily shown to be a linear function of τ at infinitesimally small τ. The TE rate $\dot{T}_{Y\to X,0} = \frac{dT_{Y\to X,\tau}}{d\tau}|_{\tau=0}$ determines the values of $T_{Y\to X,\tau}$ over the interval of small-enough τ to some characteristic time. The TE rate has the dimension of inverse time and so its numerical values depend on the time unit. So a dimensionless quantifier $T_{Y\to X} = \tau_{\text{char}}\dot{T}_{Y\to X,0}$ is used below, where τ_{char} is defined as the minimum of the individual characteristic times of the subsystems X and Y. It is called here the "reduced" original TE. The reduced infinite-history TE is defined in the same manner, after excluding a dependence of $T^{(\infty,\infty)}_{Y\to X,\tau}$ on Δt at each τ via taking first the limit of $\Delta t \to 0$ as in Ref. [71]. In practice, it corresponds just to Δt much smaller than any characteristic time of the dynamics. So we define the infinite-history TE rate $\dot{T}^{(\infty,\infty)}_{Y\to X,0} = \frac{dT^{(\infty,\infty)}_{Y\to X,\tau|\Delta t\to 0}}{d\tau}|_{\tau=0}$ and the reduced infinite-history TE $T^{\infty}_{Y\to X} = \tau_{\text{char}}\dot{T}^{(\infty,\infty)}_{Y\to X,0}$. If any TE is computed for $\tau \ll \tau_{\text{char}}$, then the respective reduced TE equals the small-τ TE multiplied by τ_{char}/τ. In the rest of the paper, the term "reduced" is usually omitted for brevity.

Note that the original TE can be expressed as a short-term DCE [43], an effect of varying an initial state $\mathbf{y}_0$ on pdf of $\mathbf{X}_\tau$, given $\mathbf{x}_0$ (Appendix B). The infinite-history TE cannot be so expressed, remaining only "directed statistical coherence" or "informational causal effect."

III. OBJECT AND METHOD

A. Reference system

Relationships between the original TE and coupling-on DCE (and infinite-history TE and noise-on DCEs as auxiliary quantities) depend on parametrization of a system. As a starting step, a complete analysis is performed here for a reasonably general class of systems which is a set of two one-dimensional linear Langevin equations describing stochastically perturbed overdamped oscillators. This is a special case of Eq. (3) given by

$$\dot{x} = -\alpha_x x + k_{xy}y + \xi_x(t), \quad \dot{y} = -\alpha_y y + k_{yx}x + \xi_y(t), \quad (6)$$

where $\alpha_x > 0$ and $\alpha_y > 0$ are relaxation rates of the subsystems and k_{xy} and k_{yx} are coupling coefficients. This simple system still has such general properties as internal dynamics of X and Y (autocorrelations), stochasticity sources (white noise), and directional couplings. Such systems are widespread in physics and other sciences as a basis for empirical modeling, e.g., approximations of large climate models [72,73].

When Eqs. (6) are derived as a conceptual model for a physical system, the quantities x, y, and t are dimensional with physical dimensions $[x]$, $[y]$, and $[t]$. In order to formulate ultimate results of this study in a more physical rather than statistical manner, appropriate dimensionless parameters are introduced below as in the theory of dimensions and similarity [60], which looks for relationships between dimensionless quantities of interest in a simple form of power laws. So define first the ratios of relaxation rates $m_{xy} = \alpha_y/\alpha_x$ and $m_{yx} = 1/m_{xy}$. Call m_{xy} *relative source rate* for the coupling $Y \to X$. The subsystem Y is the *source* of this coupling and the subsystem X is its *recipient*. If $m_{xy} > 1$ ($m_{xy} < 1$), then the coupling $Y \to X$ is *coupling from the fast (slow) source*. Call $M = \max\{m_{xy}, m_{yx}\} \geqslant 1$ the *rate difference*.

Second, in order to introduce coupling parameters, note that the variances of x and y for uncoupled systems are $\sigma^2_{x,0} = \Gamma_{xx}/(2\alpha_x)$ and $\sigma^2_{y,0} = \Gamma_{yy}/(2\alpha_y)$. Consider variances of the first and the second terms on the right-hand side of the first equation (6) in the open-loop regime, i.e., when both x and y evolve as if they were uncoupled. The ratio of the variance of the second (coupling) term to that of the first (individual) term is $\beta^2_{xy} = \frac{k^2_{xy}\sigma^2_{y,0}}{\alpha^2_x\sigma^2_{x,0}} = \frac{k^2_{xy}\Gamma_{yy}}{\alpha_x\alpha_y\Gamma_{xx}}$. This is the simplest dimensionless parameter of the coupling $Y \to X$. Relying only on the evolution equation coefficients and the "free" variances of x and y, it characterizes the coupling *immediately*, i.e., without taking into account coupled behavior. Therefore, call β^2_{xy} *immediate coupling strength* or simply *coupling strength*. If $\beta^2_{xy} > 1$ ($\beta^2_{xy} < 1$), then the coupling $Y \to X$ is *(immediately) strong* (*weak*). Geometric mean of the coupling strengths $\beta^2 = \sqrt{\beta^2_{xy}\beta^2_{yx}} = \frac{|k_{xy}k_{yx}|}{\alpha_x\alpha_y}$ is *mean coupling strength*. Denote $s = \text{sgn}\{k_{xy}k_{yx}\}$: $s = 1$ ($s = -1$) corresponds to *positive (negative) feedback*.

Third, ratios of coupling strengths are useful to characterize "predominance." Thus, call $l_{xy} = |\beta_{xy}/\beta_{yx}|$ *predominance parameter* of the coupling $Y \to X$. If $l_{xy} > 1$ ($l_{xy} < 1$), then the coupling $Y \to X$ is *predominant* (*deficient*). Everything is the same for $l_{yx} = 1/l_{xy}$. Parameter $L = \max\{l_{xy}, l_{yx}\}$ quantifies *immediate coupling difference*. If $L = 1$, then couplings are *immediately equivalent*.

Finally, it is convenient to introduce relative coupling parameters as those divided by the relative source rate. The ratio β^2_{xy}/m_{xy} is *relative coupling strength*. If $\beta^2_{xy}/m_{xy} > 1$ ($\beta^2_{xy}/m_{xy} < 1$), then the coupling $Y \to X$ is *relatively strong* (*relatively weak*). If $\beta^2_{xy} > M$ ($1 < \beta^2_{xy} < M$), then the coupling $Y \to X$ is *essentially strong* (*moderately strong*). Similarly, if $\beta^2_{xy} < 1/M$ ($1/M < \beta^2_{xy} < 1$), the coupling $Y \to X$

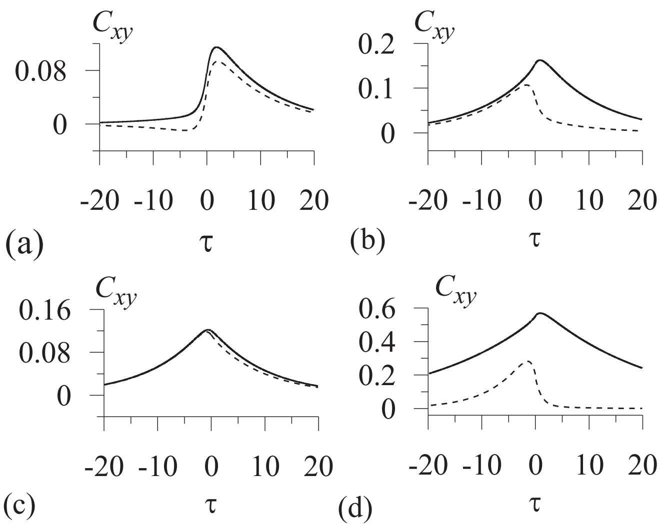

FIG. 1. CCFs for the system (6) with $m_{xy} = 10$ and various other parameters: (a) $l_{xy} = 50$, $\beta^2 = 0.01$; (b) $l_{xy} = 3$, $\beta^2 = 0.05$; (c) $l_{xy} = 0.3$, $\beta^2 = 0.005$; (d) $l_{xy} = 3$, $\beta^2 = 0.5$. Solid lines show the case of positive feedback $s = 1$, the dashed lines show the case of $s = -1$. Time units along the abscissa axes correspond to $\alpha_x = 0.1$ and $\alpha_y = 1$.

is *essentially weak* (*moderately weak*). Next, the ratio l_{xy}/m_{xy} is *relative predominance parameter* of the coupling $Y \to X$. If $l_{xy}/m_{xy} > 1$ ($l_{xy}/m_{xy} < 1$), then the coupling $Y \to X$ is *relatively predominant* (*relatively deficient*). If $l_{xy} > M$ ($1 < l_{xy} < M$), then the coupling $Y \to X$ is *essentially predominant* (*moderately predominant*). If $l_{xy} < 1/M$ ($1/M < l_{xy} < 1$), then the coupling $Y \to X$ is *essentially deficient* (*moderately deficient*). A parameter $L_r = \max\{l_{xy}/m_{xy}, l_{yx}/m_{yx}\}$ is *relative coupling difference*. If $L_r = 1$, then the couplings are *relatively equivalent*. If $L > M$ ($1 < L < M$), then the couplings are *essentially different* (*moderately different*).

B. Correlation functions

A usual characterization of stationary Gaussian processes [74,75] involves autocorrelation functions (ACFs) and cross-correlation function (CCF). Since ACFs and CCF can be used in practical estimation of DCEs from TEs (Sec. V), they are briefly commented here for some characteristic parameter values for the system (6). For sufficiently weak couplings $\beta_{xy}^2 \ll 1$ and $\beta_{yx}^2 \ll 1$, ACFs of the processes X_t and Y_t are almost exponentially decaying with decay rates equal to α_x and α_y, respectively. Figure 1 shows plots of the CCF $C_{xy}(\tau) = \langle X_t Y_{t-\tau}/(\sigma_x\sigma_y)\rangle$ computed from Eq. (C6) (Appendix C).

For essentially predominant coupling from the fast source $Y \to X$, the CCF looks quite asymmetric for any feedback sign [Fig. 1(a)], with an interval of sign reversal for negative feedback. Figure 1(b) illustrates moderately predominant (i.e., relatively deficient) coupling from the fast source $Y \to X$ and small $\beta^2 = 0.05$: Positive feedback exhibits less asymmetric form than that in Fig. 1(a), while peculiarity of the negative feedback is that the CCF maximum is attained at negative τ where X (a source of deficient coupling) leads in time, in contrast to other cases where at the CCF maximum point a source of predominant coupling leads. Figure 1(d) exhibits CCF plots similar to Fig. 1(b), though mean coupling is 10 times stronger (so CCF values are greater). Predominant coupling from the slow source in Fig. 1(c) shows almost symmetric (relatively to the maximum point) CCF plots. Thus, using the CCF plot, one can guess in practice the situations of essentially predominant coupling from the fast source (including feedback sign) and moderately predominant coupling from the fast source with negative feedback.

To distinguish other situations, one should perform an additional analysis (Appendix C). In general, ACFs and CCF are determined by the characteristic exponents $z_{1,2}$ which are the roots of the characteristic polynomial $z^2 + (\alpha_x + \alpha_y)z + \Delta = 0$, where $\Delta = \alpha_x\alpha_y - k_{xy}k_{yx}$ is the system determinant. Denote uncoupled system determinant $\Delta_0 = \alpha_x\alpha_y$, relative determinant $\tilde{\Delta} = \Delta/\Delta_0 = 1 - s\beta^2$, and inverse system determinant $1/\tilde{\Delta}$.

C. Expressions for coupling quantifiers

Recalling that stationary variance is used as the characteristic Q to define the long-term DCEs, the latter are found by solving linear algebraic equations for the stationary covariance matrix of the process (X_t, Y_t) given in Appendix C. The results read

$$S_{Y\to X} = \frac{(l_{xy} + m_{xy}s)\beta^2}{\tilde{\Delta}(1 + m_{xy})}, \tag{7}$$

and

$$N_{Y\to X} = \frac{l_{xy}\beta^2}{\tilde{\Delta} + m_{xy}}. \tag{8}$$

Reduced TEs are defined with $\tau_{\text{char}} = 1/\max\{\alpha_x, \alpha_y\}$. The original TE is found from linear ordinary differential equations for conditional moments (Appendix C) as

$$T_{Y\to X} = \frac{l_{xy}\beta^2}{4\max\{1, m_{xy}\}}(1 + S_{X\to Y})(1 - r^2), \tag{9}$$

where r is zero-lag correlation coefficient. The infinite-history TE is found via cross-spectral matrix factorization (Appendix C) and reads

$$T^{\infty}_{Y\to X} = \left(\sqrt{1 + \frac{l_{xy}\beta^2}{m_{xy}}} - 1\right)\frac{\min\{1, m_{xy}\}}{2}. \tag{10}$$

In power-law relationships of the theory of dimensions and similarity [60], their exponents and domains of applicability are of utmost importance, while numerical coefficients are typically of the order of unity and their exact values are often not so necessary, e.g., twice as small or twice as large estimates are quite acceptable. Here, relationships between the coupling quantifiers and the dimensionless parameters are also found as intermediate asymptotic in the approximate power-law form. Namely, one takes, e.g., $m_{xy} \gg 1$ and $l_{xy} \gg m_{xy}$ (or other strong inequalities) to expand Eqs. (7)–(10) into Taylor series with respect to small parameters $1/m_{xy}$ and m_{xy}/l_{xy} (or others) and retain the lowest order. The resulting relationships often remain reasonably accurate if the asymptotic conditions are violated, as specially studied below.

IV. RESULTS

A. "Four-TE" relationship

Intermediately asymptotic cases imply $L \gg 1, M \gg 1$, and $L_r \gg 1$ ($L \gg M$ or $L \ll M$) as studied in this subsection. Start with uncoupled subsystems X and Y and increase coupling strengths from zero. While mean coupling strength remains small $\beta^2 \ll 1$, coupling-on DCEs rise in absolute value proportionally to the respective coupling strength. The DCE (7) in the direction of relatively predominant coupling (say, $Y \to X$) reads [76]

$$S_{Y\to X} \approx \frac{\beta_{xy}^2}{\max\{1, m_{xy}\}}. \quad (11)$$

If this coupling is from the fast source, then this DCE equals the relative coupling strength. Otherwise, it equals the immediate coupling strength. The TE (9) reads $T_{Y\to X} \approx \beta_{xy}^2/(4\max\{1, m_{xy}\})$ for small-enough coupling strengths, when r^2 and $S_{X\to Y}$ are small. Hence, $S_{Y\to X}$ relates to $T_{Y\to X}$ for relatively predominant coupling $Y \to X$ as

$$S_{Y\to X} \approx 4T_{Y\to X}, \quad (12)$$

showing that a nats of the TE correspond to $4a$ r.u. of the coupling-on DCE. This simple relationship is very convenient for the DCE estimation and TE interpretation. The domain of its applicability is as follows.

For positive feedback, a sufficient condition for validity of the "four-TE" relationship (12) is that the coupling strengths are appropriately bounded:

(i) $\beta^2 \ll 1$, if the coupling $Y \to X$ is from the fast source and essentially predominant ($l_{xy} \gg M$);

(ii) $\beta_{xy}^2 \ll 1$, if the coupling $Y \to X$ is from the slow source and predominant ($l_{xy} \gg 1$);

(iii) $1 - \beta^2 \gg 1/L$, if the coupling $Y \to X$ is from the slow source and moderately deficient ($1/M \ll l_{xy} \ll 1$).

In case (i), the coupling $Y \to X$ can be of any strength β_{xy}^2 and the corresponding TE and DCE can be arbitrarily large. The four-TE relationship is violated for nonsmall β^2, since the relative determinant $\tilde{\Delta}$ gets considerably less than unity and a large multiplier $1/\tilde{\Delta}$ appears in the expression for $S_{Y\to X}$ (7), so the latter becomes much greater than the four-TE level $4T_{Y\to X}$. In cases (ii) and (iii), both TE and DCE are necessarily weak. The relationship (12) is violated for nonsmall β_{xy}^2 in case (ii) and β^2 exceeding $1 - 1/L$ in case (iii), since then the cross correlation r gets of the order of unity and a small multiplier $1 - r^2$ in Eq. (9) decreases the TE, making $4T_{Y\to X} \ll S_{Y\to X}$. In those ranges, $S_{Y\to X}$ rises as β_{xy}^2 in case (ii) and as $1/(L\tilde{\Delta})$ in case (iii), while the TE intermediately stabilizes at $T_{Y\to X} = 1/4$ nats.

For absent feedback, i.e., for a unidirectional coupling $Y \to X$, the condition (i) transforms to "any β_{xy}^2" since then $\beta^2 = 0$ and cross correlation is small $r^2 \ll 1$. Hence, the four-TE law (12) applies to any unidirectional coupling from the fast source. The condition (ii) remains. For negative feedback, all conditions remain the same as for positive one, only the reason for the violation of (12) at nonsmall β^2 in case (i) differs: $S_{Y\to X}$ saturates at L/M due to large $\tilde{\Delta}$ in the denominator of the right-hand side of Eq. (7) for $\beta^2 \gg 1$, while $4T_{Y\to X}$ rises as $S_{Y\to X}\beta^2$.

In total, the four-TE relationship is sufficiently widely applicable (see also Appendix D) and not even restricted to both weak couplings. Moreover, when it is not accurate up to a small relative error ($\ll$1), it may still remain reasonably accurate as studied in the next subsection. As for the relatively deficient coupling $X \to Y$, the "$S - T$" relationship differs from Eq. (12) involving an additional large factor: (i) $|S_{X\to Y}| \approx 4(L/M)T_{X\to Y}$, (ii) $|S_{X\to Y}| \approx 4(LM)T_{X\to Y}$, and (iii) $|S_{X\to Y}| \approx 4(M/L)T_{X\to Y}$.

The infinite-history TE $T_{Y\to X}^{\infty}$, often reported in published works, is close to the original TE $T_{Y\to X}$ for weak-enough couplings. Their closeness is violated if the relative coupling strength β_{xy}^2/m_{xy} is not small (i.e., not much less than unity) or the opposite DCE $S_{X\to Y}$ is so large that the factor $(1 + S_{X\to Y})(1 - r^2) - 1$ in Eq. (9) is not small. Thus, the four-TE relationship often extends to the infinite-history TE as well but not in all cases.

B. How many nats provide unit long-term effect?

A unit DCE corresponds to doubling of the recipient variance and represents quite considerable coupling role rather than very small coupling strengths. Questions of practical interest are as follows: How many nats of the original TE provide unit coupling-on DCE in the same direction? and Are such threshold values close to the four-TE law (12)?

The thresholds are shown in Fig. 2 for various couplings with dominating role in dynamics. To compute the threshold $T_{Y\to X}$, the value of $S_{Y\to X} = 1$ is substituted to Eq. (7) and the threshold β^2 is found, since both l_{xy} and m_{xy} are known for each point in the plots of Fig. 2. Having β^2, the opposite DCE $S_{X\to Y}$ is found from the same equation as (7) and r^2 from Eq. (C4). Then the threshold TE $T_{Y\to X}$ is found from Eq. (9). Figure 2(a) corresponds to the coupling $Y \to X$ from the fast source. In the unidirectional case, the threshold $T_{Y\to X}$ (solid line) is about $1/4$ nats for $m_{xy} \geqslant 10$, meeting the four-TE law. For equal rates $m_{xy} = 1$, it differs almost twice (to $3/8$ nats) which is not a very strong distinction. If the coupling $Y \to X$ is essentially predominant (long dashes for $l_{xy} = 100$), then the threshold $T_{Y\to X}$ even for not very small ratio $m_{xy}/l_{xy} \approx 0.2$ is about 0.2 nats, close to $1/4$ nats for a unidirectional coupling. For a boundary situation of relatively equivalent couplings (short dashes), the threshold is $T_{Y\to X} \approx 0.1$ nats for any m_{xy}, which differs from $1/4$ nats about twice, i.e., again not drastically.

As for the coupling from the slow source [Fig. 2(b)], the threshold $T_{X\to Y}$ in a unidirectional case varies in a wider range from $3/8$ nats (for $m_{xy} = 1$) to $1/8$ nats (for $m_{xy} \approx 100$) as shown by the solid line. This range slightly changes to (0.12, 0.3) nats for a predominant coupling $X \to Y$ with $l_{yx} = 10$ (long dashes). For a boundary situation of immediately equivalent couplings (short dashes), the threshold is exactly $T_{X\to Y} = 1/8$ nats, differing from the level of $1/4$ nats twice. Thus, a predominant or immediately equivalent coupling from the slow source will lead in practice to a smaller accuracy of the four-TE law in estimating coupling-on DCEs of the order of unity. However, even in the worst cases here, the four-TE law gives a reasonable rough estimate, twice as small or twice as large as a true value.

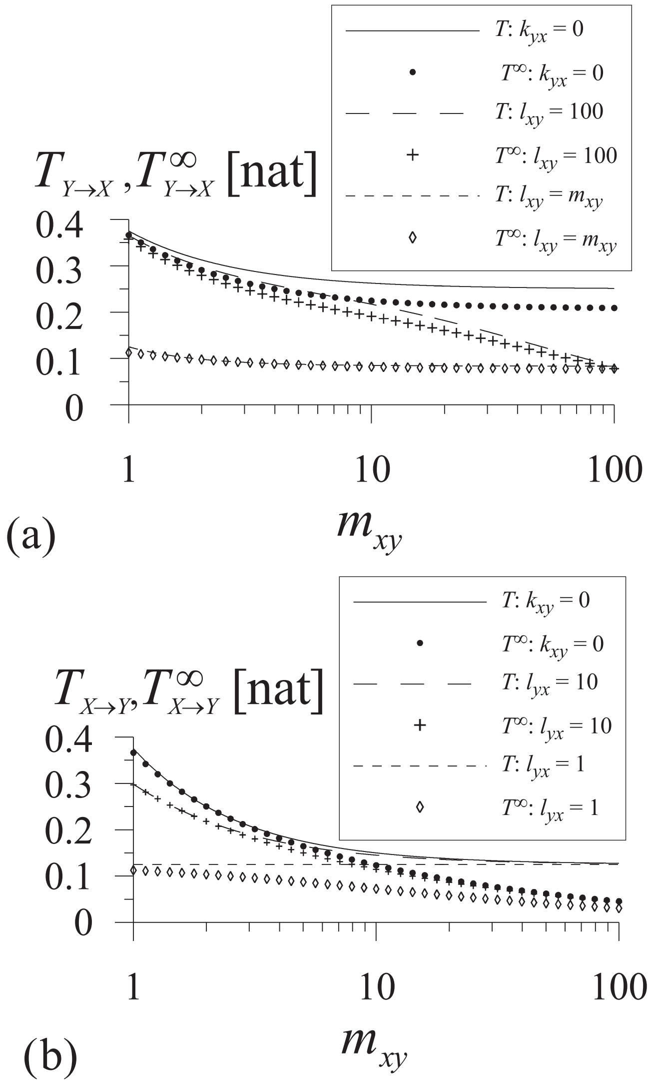

FIG. 2. Threshold values of the original and infinite-history TEs corresponding to the unit coupling-on DCE in the same direction for the system (6): (a) for the coupling from the fast source Y; (b) for the coupling from the slow source X. Parameters are given in the legends.

The threshold infinite-history TE differs from the respective original TE if the unit coupling-on DCE is achieved at large-enough (of the order of unity) relative coupling strength in that direction. Namely, the thresholds $T_{Y\to X}$ and $T^{\infty}_{Y\to X}$ in Fig. 2(a) moderately differ ($T^{\infty}_{Y\to X}/T_{Y\to X} \approx 0.8$) for unidirectional couplings and large m_{xy}, where they correspond to $\beta^2_{xy}/m_{xy} \approx 1$. In Fig. 2(b) their difference occurs for $m_{xy} \gg 1$ with threshold values of $\beta^2_{yx} \approx 1/m_{xy}$, where $T^{\infty}_{X\to Y} \approx 1/\sqrt{8m_{xy}}$ nats, $T_{X\to Y} \approx 1/8$ nats, and $T^{\infty}_{X\to Y} \approx \sqrt{8/m_{xy}}T_{X\to Y}$. The latter formula indicates that the infinite-history TE may quite strongly differ from the original TE and be a bad proxy as further addressed in the next subsection.

In total, the most typical values of the original TE providing a unit coupling-on DCE often correspond (reasonably) accurately to the four-TE law, i.e., are close to 1/4 nats varying roughly from 0.1 to 0.35 nats around this basic value. The respective infinite-history TE is sometimes close and sometimes considerably less. It is now easy to see the meaning of 1 nat. Such original TE means a very influential coupling: for a unidirectional coupling from the fast source $Y \to X$, Eqs. (7) and (9) give that the respective DCE is $S_{Y\to X} = 4$ r.u., i.e., a fivefold increase of the recipient variance. Further, $T^{\infty}_{Y\to X} = 1$ nat corresponds to even greater $S_{Y\to X} = 8$ r.u. For a unidirectional coupling from the slow source $X \to Y$, both $T_{X\to Y} = 1$ nat and $T^{\infty}_{X\to Y} = 1$ nat means $S_{X\to Y} = 4m_{xy} \gg 1$ r.u., i.e., quite large long-term DCE.

C. Qualitative distinction between TEs

The limit case of essential predominance $l_{xy} \to \infty$ is met above for a unidirectional coupling $Y \to X$, where $\beta^2_{yx} = \beta^2 = 0$ due to $k_{yx} = 0$. However, it is also met for zero noise in the coupling recipient $\Gamma_{xx} \to 0$. Then $\beta^2_{yx} \to 0$ but the mean coupling strength $\beta^2 = |k_{xy}k_{yx}|/(\alpha_x\alpha_y)$ is finite. As shown below, the original and infinite-history TEs in the direction of essentially deficient coupling behave then qualitatively differently, one being finite and another arbitrarily small.

Consider essentially strong coupling from the fast source $Y \to X$, when $\beta^2 \approx 1$ and cross correlation r is small: $1/M \ll 1 - \beta^2 \ll 1$ implies $r^2 = 1/(M\tilde{\Delta}) \ll 1$ and $l_{xy} = L \gg M$ (Table II in Appendix E). Nothing new happens in the direction of essentially predominant coupling $Y \to X$, only the coupling-on DCE becomes even greater $S_{Y\to X} = L/(M\tilde{\Delta}) \gg N_{Y\to X} = 4T_{Y\to X} = L/M$. In the direction of essentially deficient coupling $X \to Y$, the coupling-on DCE satisfies the four-TE law $S_{X\to Y} = 4T_{X\to Y} = 1/(M\tilde{\Delta}) \ll 1$, while the infinite-history TE and the noise-on DCE are much smaller $N_{X\to Y} = (4/\tilde{\Delta})T^{\infty}_{X\to Y} = 1/(L\tilde{\Delta})$. For $L \to \infty$ occurring under $\Gamma_{xx} \to 0$, both $N_{X\to Y}$ and $T^{\infty}_{X\to Y}$ tend to zero remaining proportional to each other, while both $S_{X\to Y}$ and $T_{X\to Y}$ remain finite and constant. Thus, the original TE appears to be a measure of the coupling-on DCE, while the infinite-history TE is a measure of the (strongly different) noise-on DCE. So one can conclude that the two TEs are qualitatively distinct here.

The situation remains similar for mean coupling strength of the order of unity but not very close to it. In particular, for $\beta^2 = 1/2$, it holds $S_{X\to Y} = 8T_{X\to Y} = 1/M \ll 1$ and $N_{X\to Y} = 8T^{\infty}_{X\to Y} = 1/L \ll 1/M$. The factor relating $S_{X\to Y}$ to $T_{X\to Y}$ is greater than in the previous case (and more distant from 4), while the factor relating $N_{X\to Y}$ to $T^{\infty}_{X\to Y}$ is smaller and closer to 4. So the divergence of the two TEs from each other appears to be a robust property, i.e., a typical situation in the space of dimensionless parameters.

This robustness is confirmed for an essentially deficient coupling from the fast source $Y \to X$ ($l_{xy} \ll 1/M$) and mean coupling strength close to unity ($1/M \ll 1 - \beta^2 \ll 1$). Then one gets very small $N_{Y\to X} = 4T^{\infty}_{Y\to X} = 1/(LM) \ll 1$, somewhat greater $T_{Y\to X} = 1/(4M^2) \ll 1$, and large $S_{Y\to X} = 1/\tilde{\Delta} \gg 1$ (Table II in Appendix E). Again, both $S_{Y\to X}$ and $T_{Y\to X}$ do not tend to zero under $L \to \infty$ and remain constant, while $N_{Y\to X}$ and $T^{\infty}_{Y\to X}$ do tend (satisfying the four-TE law in their turn). Thus, the original and infinite-history TEs necessarily diverge from each other for an essentially deficient coupling with nonsmall mean coupling strength and positive feedback.

D. Other relationships

All relationships between the coupling quantifiers are diverse, with the most representative and practically relevant ones presented above. Details of the full set of cases is given in Appendices D and E, while its further characteristic features are briefly commented below.

The coupling-on DCE (11) is positive independently of the feedback sign, i.e., relatively predominant coupling always increases the recipient variance. The coupling-on DCE in the opposite direction $S_{X\to Y}$ has the sign s of the feedback. As for the absolute values, the DCE in the direction of predominant coupling is exactly L times greater $S_{Y\to X}/S_{X\to Y} = l_{xy}s$. Ratios of other coupling quantifiers in the opposite directions are not so universal, e.g., $T_{Y\to X}/T_{X\to Y} \approx l_{xy}^2/m_{xy}$ for weak-enough couplings (Appendix D). Negative feedback and large cross correlation $1 - r^2 \ll 1$ are also discussed in Appendix D.

Finally, consider what happens if very strong inequalities (e.g., $L \gg 1$, $\beta^2 \ll 1/L$, etc.) are met only in a moderate sense, e.g., $\beta^2 = 3/L$ instead of $\beta^2 \gg 1/L$. As reported in Table III (Appendix E), the intermediately asymptotic "strong-inequality" relationships remain reasonably accurate, typically with relative error less than 50% and often considerably smaller. Moreover, these relationships often apply to the very boundary points (e.g., $\beta^2 = 1/L$) where they typically give an approximate value of a DCE not worse than twice as small or twice as large as a true value. A particular boundary is the case of equal rates $M = 1$ (Table IV, Appendix E), where relationships obtained for a large rate difference often apply with slight correction. Thus, for positive feedback and $\beta^2 \ll 1/L$, one gets $S_{Y\to X} = 2T_{Y\to X}$ (i.e., the two-TE relationship) instead of $S_{Y\to X} = 4T_{Y\to X}$. To generalize both cases, Eqs. (7) and (9) give $S_{Y\to X} = kT_{Y\to X}$ with $k = 4/(1 + 1/M)$ for an arbitrary M and weak-enough couplings.

The situation can get quite specific for negative feedback. Then a boundary case of relatively equivalent couplings always provides zero coupling-on DCEs in both directions (and $r = 0$), even for arbitrarily large coupling strengths. This is not reflected by the TEs which get then arbitrarily large as well. This is an illustration that a simple relationship like the four-TE law is not sometimes relevant even as an approximation and, in addition to TEs, more detailed description of dynamics is needed to learn dynamical effects of directional couplings.

V. DISCUSSION

The task to find the long-term DCEs from estimates of TEs, ACFs, and CCF readily arises, e.g., in meta-analysis studies, where published works report resulting figures for those quantities without the values of empirical model coefficients. Then one cannot use a full model to estimate the coupling-on DCEs directly from their definition (4) by manipulating model coefficients [77]. So relationships between TEs and DCEs can be applied to estimate the latter. Knowing ACFs and CCF, one can guess where the system is in the space of dimensionless parameters and choose relationships to be used to determine $S_{Y\to X}$ from $T_{Y\to X}$. Below, let us focus on application of the simplest and sufficiently general four-TE law, while more complicated cases are discussed in Appendix F.

First, one should consider sample ACFs of both processes under study and verify that their plots are close to exponential decay. In addition, the CCF maximum over time lags should be small enough, e.g., $\max_\tau r^2(\tau) < 0.1$ suffices. Then each relaxation rate equals the respective ACF decay rate. Thereby, one gets also the value of m_{xy}. For all that, the sampling interval should be small-enough $\Delta t \ll \tau_{\text{char}} = 1/\max\{\alpha_x, \alpha_y\}$, e.g., $\Delta t \leqslant 0.2\tau_{\text{char}}$.

Second, for this small Δt, the ordinary one-step-ahead TEs multiplied by $1/(\max\{\alpha_x, \alpha_y\}\Delta t)$ provide the respective reduced TEs. Only for an essentially strong coupling, a maximal value of an ordinary τ-dependent TE can be achieved at $\tau_{\max} \ll \tau_{\text{char}}$, so one needs even smaller $\Delta t \leqslant 0.2\tau_{\max}$ to estimate a reduced TE reliably. This condition is met if an ordinary TE versus τ is linear with zero intercept within the range of the smallest available $\tau = \Delta t, 2\Delta t, \ldots,$ and $\tau_{\max} \geqslant 5\Delta t$. As for the TE estimation, recall that for a two-dimensional stationary Gaussian process (X_t, Y_t) and small τ, the TE simply relates to the respective prediction improvement [49], so $T_{Y\to X} = (1/2)G_{Y\to X}$ (Appendix A). One can use an estimate of WG causality often reported in the literature to assess the TE, if the latter is not given.

Third, if couplings in both directions are nonzero, then an estimate of predominance parameter is found as $l_{xy} = \sqrt{m_{xy}T_{Y\to X}/T_{X\to Y}}$. Depending on the value of l_{xy}/m_{xy}, one identifies direction of relatively predominant coupling and the four-TE law gives the coupling-on DCE for that direction. The relationship with a factor less than 4, namely $k = 4/(1 + 1/M)$, can be used if the rates are not very different. The coupling-on DCE in the opposite direction is given by the corrected four-TE (or k-TE) law with an additional large factor (either L_r or LM, Sec. IV A) and its sign s can be often determined from the CCF plot. Having the DCE estimates, one can check again whether conditions for applicability of those relationships are still fulfilled.

If CCF values are not small enough, then the observed ACF decay rates may differ from α_x and α_y, depending on other parameters of the system. Other conditions for applicability of the four-TE law may not be fulfilled as well, e.g., mean coupling strength β^2 may not be small. Then one should either use prior knowledge of some system parameters (or full empirical model for direct estimation of the DCEs) or identify which of diverse relationships applies, using the four Tables in Appendix E.

Details of application procedure deserve a careful statistical study. Here consider a simple illustration of how the suggested relationships allow one to have an intuitive "feeling" of the TE numerics while reading reported results on WG causality or TE analysis. References [78,79] provide estimates of the WG causality from solar activity variations (Y) on tropical Atlantic climate (X) from paleoclimate time series (total solar irradiance from combined paleoclimate archives and stalagmite $\delta^{18}O$ data at Yok Balum cave, southern Belize) over the past 2000 years at a time step of $\Delta t = 5$ years. After coping carefully with dating errors, the authors have revealed statistically significantly (at $p < 0.05$) a unidirectional influence of solar activity variations on the regional rainfall variations with (optimally selected) one-step-ahead relative prediction improvement $G^{(4,1)}_{Y\to X,\Delta t} = 0.015$ r.u. As the plots in supplemental material [79] show [Figs. S1(b) and S2(b)], ACFs of both signals are close to exponential decay with $1/\alpha_x = 20$ years and $1/\alpha_y = 25$ years. Maximal CCF is about 0.3 [Fig. S4(b)]. The τ-dependent PI $G^{(4,1)}_{Y\to X,\tau}$ rises approximately linearly over small $\tau = \Delta t, 2\Delta t, 3\Delta t$ from 0.015 to $\approx$0.04 r.u. and seemingly reaches maximum at $\tau = 4\Delta t$ [Fig. S6(a)]. Thus, conditions for a weak unidirectional coupling in a system (6) are fulfilled. The reduced TE (either original or infinite-history) equals $G^{(4,1)}_{Y\to X,\Delta t}/(2\alpha_x\Delta t) \approx 0.03$ nats. Since the relaxation rates ratio is $m_{xy} \approx 0.8$, the two-TE relationship can be used to estimate $S_{Y\to X}$ [or, more accurately, the k-TE relationship with $k = 4/(1 + m_{xy}) \approx 2.2$]. Thus, one gets $S_{Y\to X} \approx$ 6–7 r.u. This is a result of a simple meta-analysis without extensive computations. To summarize, the published values of WG causality are found to correspond to the reduced TE from solar activity variations to the Atlantic hydroclimate of 0.03 nat and to the increase of the recipient (Belizean rainfall) variance by 6–7% due to the solar influence. Further studies seem to be in order to understand how significant this coupling is from the physical viewpoint and whether it is a manifestation of a more global solar influence on the Earth climate. In particular, a moving-window analysis [79] has shown that in the first millennium A.D. the relative PI is about three times as great and so may correspond to about 20% increase of the recipient variance due to solar influence, but reliability of the estimates from such shorter time windows is lower. Diverse examples of one-step-ahead relative PIs of one to several hundredth in climate data (e.g., Refs. [80–82]) can readily get similar DCE interpretations.

If a practical situation does not resemble two one-dimensional systems (6), then one can construct an empirical autoregressive model from a time series and estimate the coupling-on DCEs directly, assuming that the model remains adequate under zeroing of the coupling coefficients. However, this study can be also extended to wider classes of systems which are of interest in practice, e.g., to higher-dimensional linear systems where one has less vivid expressions for the TEs via generalized variances [49] and can expect certain multi-dimensional (matrix) generalizations of the relationships (7)–(10) to hold.

For nonlinear systems, long-term DCEs may be more appropriately characterized with other quantities in addition to variance. Still, some classes of systems, e.g., weak nonlinearity in the presence of noise or simple unimodal stationary pdf determined by sufficiently strong noise, may well lead to S-T relationships close to those obtained here. In such situations one can also expect the original and infinite-history TEs to be sometimes close to each other and sometimes diverging to different long-term DCEs. One can expect the infinite-history TE to be related to responses of the recipient ACF and power spectral density to switching the source noise on. The original TE is in any case expected to be closely related to the coupling-on DCE in terms of some appropriate stationary characteristic Q in Eq. (4). Further studies of relationships between various quantifiers of directional couplings for reasonably wide and practically interesting classes of stochastic dynamical systems seem to promise useful results. As such quantifiers, one can consider phase dynamics-based characteristics [83–90], nonlinear Granger causality measures [91–94], convergent cross mapping [53,54,95] and similar state space measures [59,96–103], "information flows" [13,14,16], empirical model-based approaches [104–108], and others, e.g., Refs. [12,109,110].

VI. CONCLUSIONS

Transfer entropy $T_{Y\to X}$ reduced to a characteristic time of a system (X, Y), a fundamental characteristic of information transfer between time-evolving objects X and Y, is related here to a coupling-on long-term dynamical causal effect $S_{Y\to X}$ which is a relative change of recipient variance under switching the coupling on. Though this relationship is diverse, depending on dimensionless parameters of a system under study (bidirectionally coupled one-dimensional linear Langevin equations), a reasonably general and practically applicable case is the four-TE law $S_{Y\to X} = 4T_{Y\to X}$ which holds for relatively predominant coupling $Y \to X$, strongly different relaxation rates, and properly bounded mean coupling strength. A more general relationship, still for the bounded mean coupling strength, is $S_{Y\to X} = kT_{Y\to X}$ with $k = 4/(1 + 1/M)$ where $M \geqslant 1$ is the rates ratio. So it becomes the two-TE law for equal rates.

These relationships and conditions of their applicability are found as intermediate asymptotic [60], allowing practical estimation of the coupling-on DCE from an estimate of the respective TE and, thereby, richer interpretation (closer "intuitive feeling") of the TE numerics. In particular, the reduced TE $T_{Y\to X}$ of 1 nat provides quite large coupling-on DCE in the same direction, ranging from 4 r.u. (fivefold increase of the recipient variance) to much greater values.

While the above original TE $T_{Y\to X}$ shows reduction of uncertainty in the future of X conditioned by an initial state, one often uses the infinite-history TE which is a measure of Wiener-Granger causality. Both TEs are shown here to be numerically close to each other under a well-defined condition of weak-enough couplings but drastically diverging from each other for an essentially deficient coupling, positive feedback, and nonsmall mean coupling strength. The infinite-history TE then tends to zero under an increase of the coupling predominance parameter $L \to \infty$ being proportional to the source noise-on DCE, while the original TE remains finite and constant being proportional to the coupling-on DCE. This should be taken into account when estimating DCEs and interpreting TEs: The widely used infinite-history TE is not always a good proxy for the original TE.

ACKNOWLEDGMENT

The work is supported by the Russian Science Foundation (Grant No. 19-12-00201).

APPENDIX A: TRANSFER ENTROPY FORMULAS AND UNITS

The (differential) Shannon entropy of a variable $\mathbf{X}$ with pdf $p_X(\mathbf{x})$ is given by $I_X = -\int p_X(\mathbf{x}) \ln p_X(\mathbf{x}) d\mathbf{x}$. Conditional entropy of $\mathbf{X}$ conditioned by $\mathbf{Y}$ is $I_{X|Y} = -\int p_{XY}(\mathbf{x}, \mathbf{y}) \ln p_{X|Y}(\mathbf{x}|\mathbf{y}) d\mathbf{x} d\mathbf{y}$, where $p_{XY}(\mathbf{x}, \mathbf{y})$ is joint pdf of $\mathbf{X}$ and $\mathbf{Y}$ and $p_{X|Y}(\mathbf{x}|\mathbf{y}) = p_{XY}(\mathbf{x}, \mathbf{y})/p_Y(\mathbf{y})$ is conditional

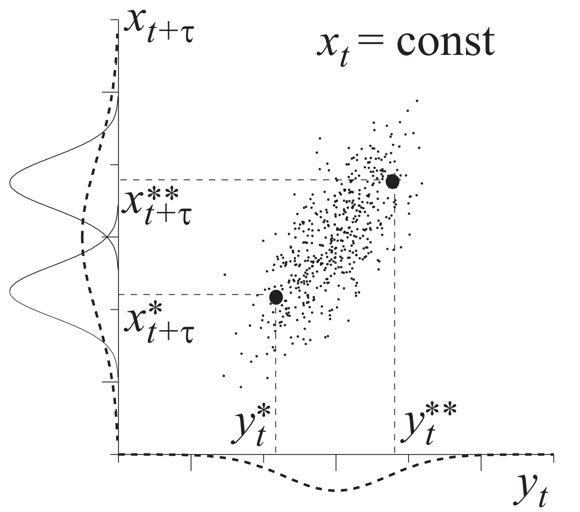

FIG. 3. Illustration of dependence between $X_{t+\tau}$ and Y_t, given $x_t = \text{const}$ and $\tau > 0$. Joint pdf of $X_{t+\tau}$ and Y_t is shown as a cloud of small circles. Marginal pdfs of Y_t and $X_{t+\tau}$ (given x_t) are shown with dashed lines along abscissa and ordinate axes, respectively. Vertical dashed lines represent sections of the pdf of $X_{t+\tau}$ under an additional condition of given y_t: Two values y^*_t and y^{**}_t produce two different conditional pdfs of $X_{t+\tau}$ shown along the ordinate axis with solid lines. The respective conditional expectations $x^*_{t+\tau}$ and $x^{**}_{t+\tau}$ are shown with large filled circles.

pdf of $\mathbf{X}$. Difference of the respective conditional entropies is the (k, l)-history TE defined in the main text as $T_{Y\to X,\tau} = I_{X,\tau|X} - I_{X,\tau|X,Y}$. It quantifies how strongly uncertainty in the current value of $\mathbf{x}$, given its k previous values, is reduced if the previous l values of $\mathbf{y}$ are taken into account.

For a two-dimensional Gaussian process (X_t, Y_t), the (k, l)-history TE relates [49] to the ratio of variances as

$$T^{(k,l)}_{Y\to X,\tau} = \frac{1}{2}\ln\frac{\operatorname{var}\{X_{t+\tau}|X^{(k)}_t\}}{\operatorname{var}\{X_{t+\tau}|X^{(k)}_t, Y^{(l)}_t\}}, \quad \text{(A1)}$$

where $\operatorname{var}\{\cdot|\cdot\}$ stands for the conditional variance. This ratio relates to the relative prediction improvement (PI) of the process x when the past of y is taken into account, since the conditional variances are just the variances of prediction errors. Denote the relative PI,

$$G^{(k,l)}_{Y\to X,\tau} = \frac{\operatorname{var}\{X_{t+\tau}|X^{(k)}_t\} - \operatorname{var}\{X_{t+\tau}|X^{(k)}_t, Y^{(l)}_t\}}{\operatorname{var}\{X_{t+\tau}|X^{(k)}_t, Y^{(l)}_t\}}, \quad \text{(A2)}$$

which is the absolute PI divided by the variance of joint prediction error. Hence, the TE is $T^{(k,l)}_{Y\to X,\tau} = \frac{1}{2}\ln[1 + G^{(k,l)}_{Y\to X,\tau}]$ [49]. Note that it differs from an ordinary normalized PI which is taken to be the absolute PI divided by the self-prediction error variance and so equals $G^{(k,l)}_{Y\to X,\tau}/[1 + G^{(k,l)}_{Y\to X,\tau}] \leqslant 1$.

The meaning of the relative PI is illustrated for $k = l = 1$ in Fig. 3, where the two-dimensional pdf of $X_{t+\tau}$ and Y_t conditioned on x_t is shown. The conditional expectation of $X_{t+\tau}$ at additionally fixed $Y_t = y^*_t$ is denoted $x^*_{t+\tau}$. Two such expectations are shown in Fig. 3 with filled circles. The variance of $X_{t+\tau}$ (i.e., of the pdf shown with the dashed line along the ordinate axis) is the conditional variance $\operatorname{var}\{X_{t+\tau}|x_t\}$ or the self-prediction error variance. It is the sum of the two terms: The variance of the conditional expectation $\operatorname{var}\{x^*_{t+\tau}\}$ (obtained via averaging over the pdf of y^*_t shown below the abscissa axis) and the conditional variance $\operatorname{var}\{X_{t+\tau}|x_t, y_t\}$ which is the joint prediction error variance (it is the variance of any conditional pdf shown along the ordinate axis with a solid line). Hence, the former term is the difference of the two prediction error variances ($\operatorname{var}\{X_{t+\tau}|x_t\} - \operatorname{var}\{X_{t+\tau}|x_t, Y_t\}$) and represents the contribution of the (conditioned on x_t) dependence between Y_t and $X_{t+\tau}$ to the self-prediction error of $X_{t+\tau}$. The relative PI is just the ratio of this contribution to the joint prediction error variance. Its units can be called relative units (r.u.). If the ordinary normalized PI equals 1/2 (or 50%), then $G^{(k,l)}_{Y\to X,\tau} = 1$ r.u., i.e., contribution of the absolute PI to the self-prediction error of $X_{t+\tau}$ equals contribution of the joint prediction error, and the TE then equals $(1/2)\ln 2 \approx 0.7$ nats.

What values of the relative PI (and TE) imply a strong coupling $Y \to X$? A simple approach is to call the relative PI large, if it exceeds 1. Then the contribution of the dependence between Y_t and $X_{t+\tau}$ to the self-prediction error of $X_{t+\tau}$ is greater than the contribution of the joint prediction error variance, i.e., greater than half the self-prediction error variance. This terminology is justified but insufficient to call the coupling itself strong or weak, since the relative PI is then large or small only in comparison with its own unit, without taking into account any features of coupled and uncoupled dynamics.

If the relative PI is much less than unity, then one gets a very simple relation $T^{(k,l)}_{Y\to X,\tau} \approx (1/2)G^{(k,l)}_{Y\to X,\tau}$, so a certain fraction of the nat corresponds to two such fractions of the r.u., e.g., the relative PI of 0.2 r.u. corresponds to the TE of 0.1 nats. In particular, it holds for infinitesimal τ and, hence, for the rates of these quantities.

The entire consideration applies to multivariate Gaussian processes if generalized conditional variances (determinants of the generalized covariance matrices) [49] are used instead of the ordinary conditional variances.

APPENDIX B: TRANSFER ENTROPY AS SHORT-TERM DYNAMICAL CAUSAL EFFECT

The original TE $T^{(1,1)}_{Y\to X,\tau}$ can be expressed as a short-term DCE as discussed in Ref. [43], where that DCE is introduced for a system (X, Y) demonstrating a Markov process (X_t, Y_t). However, the viewpoint is changed to the "intervention-effect" perspective [47]. Instead of asking how the future is predicted based on the observed past, one looks at how the future responses to (independent) variations of the current state. Then it is no longer compulsory to restrict consideration with a stationary regime of (X, Y). Rather, one relies on an explicit description of the system under study as a stochastic dynamical system, whose initial state $(\mathbf{x}_0, \mathbf{y}_0)$ uniquely determines all future conditional pdfs. The evolution is specified formally as $p_{X,Y,\tau|X,Y}(\mathbf{x}_\tau, \mathbf{y}_\tau|\mathbf{x}_0, \mathbf{y}_0) = L_{X,Y,\tau}(\mathbf{x}_0, \mathbf{y}_0)$, where the operator $L_{X,Y,\tau}$ uniquely relates an initial state $(\mathbf{x}_0, \mathbf{y}_0)$ to a conditional pdf of $(\mathbf{X}_\tau, \mathbf{Y}_\tau)$ at $\tau > 0$ and serves as a transition pdf. Through marginalization of the image of $L_{X,Y,\tau}(\mathbf{x}_0, \mathbf{y}_0)$ over $\mathbf{y}$ or $\mathbf{x}$, one gets the operators $L_{X,\tau}(\mathbf{x}_0, \mathbf{y}_0)$ or $L_{Y,\tau}(\mathbf{x}_0, \mathbf{y}_0)$, which uniquely relate an initial state to marginal pdfs of $\mathbf{X}_\tau$

and $\mathbf{Y}_\tau$, respectively:

$$p_{X,\tau|X,Y}(\mathbf{x}_\tau|\mathbf{x}_0,\mathbf{y}_0) = L_{X,\tau}(\mathbf{x}_0,\mathbf{y}_0),$$
$$p_{Y,\tau|X,Y}(\mathbf{y}_\tau|\mathbf{x}_0,\mathbf{y}_0) = L_{Y,\tau}(\mathbf{x}_0,\mathbf{y}_0). \quad \text{(B1)}$$

Here Y does not affect X if and only if $\partial L_{X,\tau}(\mathbf{x}_0,\mathbf{y}_0)/\partial\mathbf{y}_0 \equiv 0$ for any $\tau > 0$. If the latter condition does not hold, then one says that the directional coupling $Y \to X$ exists. Everything is symmetric for the direction $X \to Y$. A representative of (B1) widespread in physics and other disciplines is given by stochastic differential equations, where the operators $L_{X,\tau}$ and $L_{Y,\tau}$ represent a solution to the Fokker-Planck equation. Under general conditions, if the system starts from an initial Dirac-delta pdf, then it converges to a certain stationary pdf after a transient process. After the convergence has occurred, the process $(\mathbf{X}_t, \mathbf{Y}_t)$ becomes stationary, while it is in general nonstationary at smaller times t.

Short-term (transient) DCEs quantify transient responses of the states of X in finite time τ to variations of the initial state of Y, given the initial state of X. An elementary short-term DCE $F_{Y\to X,\tau}(\mathbf{x}_0,\mathbf{y}_0^*,\mathbf{y}_0^{**})$ quantifies the difference between the conditional pdfs $p_{X,\tau|X,Y}(\mathbf{x}|\mathbf{x}_0,\mathbf{y}_0^*)$ and $p_{X,\tau|X,Y}(\mathbf{x}|\mathbf{x}_0,\mathbf{y}_0^{**})$ using the Kullback-Leibler divergence $D(p||q) = \int p(x)\ln(p(x)/q(x))dx$. Via averaging over $(\mathbf{x}_0,\mathbf{y}_0^*,\mathbf{y}_0^{**})$ with pdf $p_X^{\text{st}}(\mathbf{x}_0)p_{Y|X}^{\text{st}}(\mathbf{y}_0^*|\mathbf{x}_0)p_{Y|X}^{\text{st}}(\mathbf{y}_0^{**}|\mathbf{x}_0)$ expressed via the stationary pdf $p_{XY}^{\text{st}}(\mathbf{x}_0,\mathbf{y}_0)$, one gets the short-term DCE $F_{Y\to X,\tau}$ depending only on the response time τ. It was shown [43] that $F_{Y\to X,\tau}$ is a sum of the original TE and an additional positive term. Hence, the original TE is a lower bound for the short-term DCE $F_{Y\to X,\tau}$. The relationship between both quantities simplifies to $T^{(1,1)}_{Y\to X,\tau} = (1/2)F_{Y\to X,\tau}$ if their values are infinitesimally small or, in practice, just much less than unity.

If the system under study is given by linear stochastic differential equations with one-dimensional state vectors x and y, then the process (X_t, Y_t) is Gaussian and the variance of the mean-squared difference between two values of x_τ obtained at fixed value of x_0 and two different values y_0^* and y_0^{**} independently sampled from the stationary pdf conditioned by x_0, is a sum of two terms. The first term is the squared difference of the conditional expectations $(x_\tau^* - x_\tau^{**})^2$ averaged over y_0^* and y_0^{**}, i.e., the response of the conditional expectation of x_τ to the variation of y_0 (Fig. 3). The second term is the doubled conditional variance of x_τ conditioned on (x_0, y_0) which is independent of a concrete value of (x_0, y_0). The short-term DCE $F_{Y\to X,\tau}$ then equals the ratio of the first term to the second term. Its units can also be called r.u. The short-term DCE of 1 r.u. means that the two contributions to the squared difference of x_τ are equal to each other. For a Gaussian process, the short-term DCE equals the relative PI: $F_{Y\to X,\tau} = G^{(1,1)}_{Y\to X,\tau}$. If $F_{Y\to X,\tau} \ll 1$, then $T^{(1,1)}_{Y\to X,\tau} = (1/2)F_{Y\to X,\tau}$, i.e., the original TE of a nats corresponds to the short-term DCE of $2a$ r.u.

APPENDIX C: DETAILS OF REFERENCE SYSTEM

Recall that the noise-free (i.e., $\boldsymbol{\Gamma} = 0$) dynamics of the basic system (6) is described by its characteristic exponents $z_{1,2}$ which are the roots of the characteristic polynomial, i.e., satisfy $z^2 + (\alpha_x + \alpha_y)z + \Delta = 0$, where $\Delta = \alpha_x\alpha_y - k_{xy}k_{yx}$ is the system determinant. These exponents read $z_{1,2} = -\frac{\alpha_x+\alpha_y}{2} \pm \sqrt{\frac{(\alpha_x+\alpha_y)^2}{4} - \Delta}$. In the uncoupled case, both noise-free subsystems demonstrate exponential decay of the initial perturbation with the respective exponent α_x or α_y, while in the presence of noise their ACFs behave so and their CCF is zero.

For positive feedback and $\Delta > 0$ (i.e., for the mean coupling strength $0 < \beta^2 < 1$), the noise-free system is stable with the fixed point of the node type. In the presence of noise, the ACFs decay slower than in the uncoupled case, i.e., power spectral densities (PSDs) become stronger concentrated at lower frequencies. The CCF becomes nonzero and the variances of x and y increase. If the mean coupling strength increases as $\beta^2 \to 1$, then the inverse system determinant rises as $1/(1-\beta^2) \to \infty$ and both variances tend to infinity as well. For $\beta^2 > 1$ the noise-free system becomes unstable and the processes x_t and y_t in the presence of noise are nonstationary.

For negative feedback and large-enough mean coupling strength $\beta^2 > (\alpha_x - \alpha_y)^2/(4\alpha_x\alpha_y)$, the noise-free system exhibits stable fixed point of the focus type. The PSDs in the presence of noise become stronger concentrated at the characteristic frequency $|\text{Im}\{z_{1,2}\}|$ of the decaying oscillations. These oscillations become visible in the ACFs plots at large-enough β^2. The variances of x and y depend on other parameters in a complicated way.

Figure 4 presents the terminology of coupling strength and predominance introduced in the main text. Note that the notions of *relatively strong* and *essentially strong* coupling coincide for the coupling from the fast source (as well as notions of *relatively predominant* and *essentially predominant* coupling). The notions of *relatively weak* and *essentially weak* coupling coincide for the coupling from the slow source (as well as notions of *relatively deficient* and *essentially deficient* coupling).

1. Original TE, DCEs, and CFs

To compute the original TE for the system (6), consider evolution of the vector $\mathbf{z}_t = (\mathbf{x}_t, \mathbf{y}_t)$. For an initial state $\mathbf{z}_0 = (x_0, y_0)$, the conditional pdf $p(x_t, y_t|\mathbf{z}_0)$ at any $t > 0$ is Gaussian with expectations $m_{x|\mathbf{z}_0}(t)$ and $m_{y|\mathbf{z}_0}(t)$, variances $\sigma^2_{x|\mathbf{z}_0}(t)$ and $\sigma^2_{y|\mathbf{z}_0}(t)$, and covariance $\sigma_{xy|\mathbf{z}_0}(t)$ given by (e.g., Ref. [43])

$$\dot m_{x|\mathbf{z}_0} = -\alpha_x m_{x|\mathbf{z}_0} + k_{xy} m_{y|\mathbf{z}_0},$$
$$\dot m_{y|\mathbf{z}_0} = -\alpha_y m_{y|\mathbf{z}_0} + k_{yx} m_{x|\mathbf{z}_0}, \quad \text{(C1)}$$

and

$$\dot\sigma^2_{x|\mathbf{z}_0} = -2\alpha_x\sigma^2_{x|\mathbf{z}_0} + 2k_{xy}\sigma_{xy|\mathbf{z}_0} + \Gamma_{xx},$$
$$\dot\sigma^2_{y|\mathbf{z}_0} = -2\alpha_y\sigma^2_{y|\mathbf{z}_0} + 2k_{yx}\sigma_{xy|\mathbf{z}_0} + \Gamma_{yy},$$
$$\dot\sigma_{xy|\mathbf{z}_0} = -(\alpha_y + \alpha_y)\sigma_{xy|\mathbf{z}_0} + k_{yx}\sigma^2_{x|\mathbf{z}_0} + k_{xy}\sigma^2_{y|\mathbf{z}_0}, \quad \text{(C2)}$$

where $\Gamma_{xy} = 0$ has been used, $m_{x|\mathbf{z}_0}(0) = x_0$, $m_{y|\mathbf{z}_0}(0) = y_0$, and $\sigma^2_{x|\mathbf{z}_0}(0) = \sigma^2_{y|\mathbf{z}_0}(0) = \sigma_{xy|\mathbf{z}_0}(0) = 0$. These are linear equations which can be solved analytically. Conditional variance $\sigma^2_{x|\mathbf{z}_0}(t)$ is the joint mean-squared prediction error, which equals $\Gamma_{xx}t$ at infinitesimal t. Conditional variance $\sigma^2_{x|x_0}(t)$ is the mean-squared self-prediction error, which at such t equals $\Gamma_{xx}t + \text{var}\{m_{x|x_0}(t)\} = \Gamma_{xx}t + k^2_{xy}\sigma^2_{y|x}t^2$, where $\sigma^2_{y|x}$ is

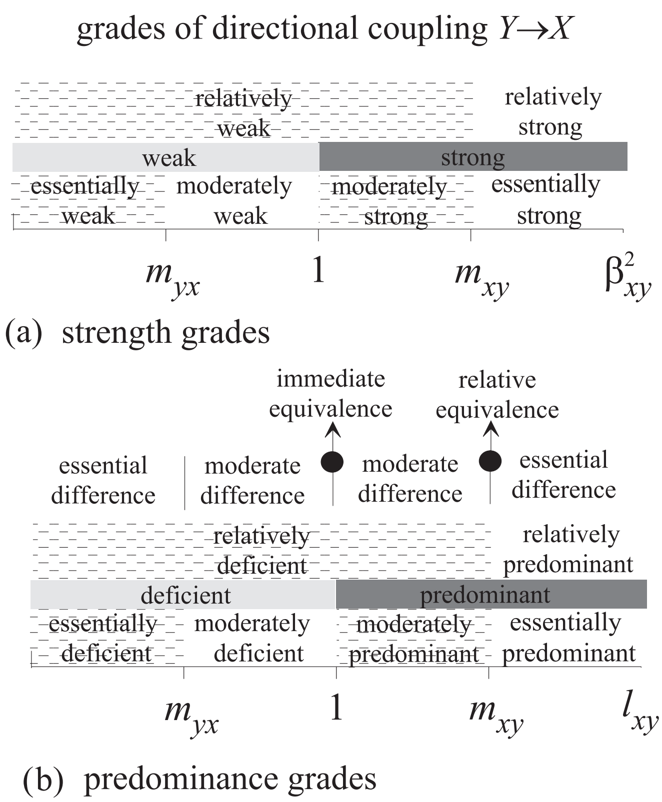

FIG. 4. Grades of the coupling $Y \to X$ for $m_{xy} > 1$ according to its (a) immediate strength and (b) predominance parameter. Logarithmic scale is used along both abscissa axes. The upper row of names in the right panel presents typical situations of difference and points of equivalence of the two directional couplings: essential difference ($L \gg M$), moderate difference ($L \ll M$), immediate equivalence ($\beta_{xy}^2 = \beta_{yx}^2$), and relative equivalence ($\beta_{xy}^2/m_{xy} = \beta_{yx}^2$).

stationary conditional variance of y equal to $\sigma_y^2(1-r^2)$ with $r = \sigma_{xy}/(\sigma_x\sigma_y)$ standing for the stationary zero-lag cross-correlation coefficient. For Gaussian processes and infinitesimal t, the original TE equals one-half of the relative prediction improvement $[\sigma^2_{x|x_0}(t) - \sigma^2_{x|\mathbf{z}_0}(t)]/\sigma^2_{x|\mathbf{z}_0}(t)$ [49]. Noting $\sigma_y^2 = \sigma_{y,0}^2(1+S_{X\to Y})$, one gets $T_{Y\to X} = k_{xy}^2\sigma_{y,0}^2(1-r^2)(1+S_{X\to Y})/(2\max\{\alpha_x,\alpha_y\}\Gamma_{xx})$. Recalling $\sigma_{y,0}^2 = \Gamma_{xx}/(2\alpha_x)$, one gets the TE in the form (9).

To derive stationary variances and covariance of X and Y which equal the above conditional moments under $t \to \infty$, one sets the left-hand side of (C2) equal to zero and solves the respective linear algebraic equations for the three stationary second-order moments. The resulting stationary covariance σ_{xy} is

$$\sigma_{xy} = \frac{\sigma_{x,0}\sigma_{y,0}(\beta_{xy} + m_{xy}\beta_{yx})}{\tilde{\Delta}(1+m_{xy})}, \tag{C3}$$

where β_{xy} and β_{yx} are signed quantities whose signs coincide with the signs of k_{xy} and k_{yx}, respectively. Then $r = \sigma_{xy}/(\sigma_x\sigma_y)$ and its squared value r^2 reads

$$r^2 = \frac{(m_{xy}l_{yx} + m_{yx}l_{xy} + 2s)\beta^2}{[1+m_{xy}+(l_{xy}-s)\beta^2][1+m_{yx}+(l_{yx}-s)\beta^2]}, \tag{C4}$$

the stationary variance σ_x^2 reads

$$\sigma_x^2 = \sigma_{x,0}^2\left[1 + \frac{(l_{xy}+m_{xy}s)\beta^2}{\tilde{\Delta}(1+m_{xy})}\right], \tag{C5}$$

and similarly for σ_y^2. Equation (C5) gives the coupling-on DCE (7) recalling $S_{Y\to X} = (\sigma_x^2 - \sigma_{x,0}^2)/\sigma_{x,0}^2$.

The (normalized) ACFs and CCF are found from the ordinary differential equations (e.g., Ref. [75]) given by

$$\begin{aligned}
dC_{xx}(\tau)/d\tau &= -\alpha_x C_{xx}(\tau) + (\sigma_y/\sigma_x)k_{xy}C_{yx}(\tau),\\
dC_{xy}(\tau)/d\tau &= -\alpha_x C_{xy}(\tau) + (\sigma_y/\sigma_x)k_{xy}C_{yy}(\tau),\\
dC_{yx}(\tau)/d\tau &= -\alpha_y C_{yx}(\tau) + (\sigma_x/\sigma_y)k_{yx}C_{xx}(\tau),\\
dC_{yy}(\tau)/d\tau &= -\alpha_y C_{yy}(\tau) + (\sigma_x/\sigma_y)k_{yx}C_{xy}(\tau),
\end{aligned} \tag{C6}$$

with $\tau \geqslant 0$, $C_{xy}(-\tau) = C_{yx}(\tau)$, and initial conditions $C_{xx}(0) = C_{yy}(0) = 1$ and $C_{xy}(0) = C_{yx}(0) = r$. The latter is found from Eq. (C3), as well as σ_x and σ_y are given by Eq. (C5) and a similar one.

For definiteness, the signs of k_{xy} and k_{yx} in the SDS (6) are selected in this work so to provide nonnegative r. Simultaneous reversal of both signs changes only the sign of each value of $C_{xy}(\tau)$. So for positive feedback, both k_{xy} and k_{yx} are positive. For negative feedback, the coupling coefficient in the direction of *relatively predominant* coupling is *positive* and the opposite one is negative.

2. Infinite-history TE

The infinite-history TE for the system (6) can be found in closed form using spectral factorization of the power spectral density matrix of the process (X_t, Y_t) similarly to a discrete-time example considered in Ref. [111]. This approach can be summarized as follows. The infinite-history TE rate equals one-half of the WG causality rate and so reads [46,71,112]

$$\dot{T}^{(\infty,\infty)}_{Y\to X,0} = \frac{1}{4\pi}\int_{-\infty}^{\infty} G_{Y\to X}(\omega)d\omega, \tag{C7}$$

where the Granger-Geweke spectrum $G_{Y\to X}(\omega)$ characterizing the coupling $Y \to X$ in the frequency domain is given for the first-order system (6) with zero noise covariance $\Gamma_{xy} = 0$ by

$$G_{Y\to X}(\omega) = \ln\left[1 + \frac{|H_{xy}(\omega)|^2 W_{yy,0}(\omega)}{W_{xx,0}(\omega)}\right], \tag{C8}$$

where $W_{xx,0}(\omega) = \frac{\Gamma_{xx}}{2\pi(\alpha_x^2+\omega^2)}$ and $W_{yy,0}(\omega) = \frac{\Gamma_{yy}}{2\pi(\alpha_y^2+\omega^2)}$ are power spectral densities of the uncoupled processes x and y, respectively, and $H_{xy}(\omega) = \frac{k_{xy}}{\alpha_x + i\omega}$ is the transfer function of the unidirectional coupling $Y \to X$. Taking the integral (C7) by parts, one gets

$$\begin{aligned}
\dot{T}^{(\infty,\infty)}_{Y\to X,0} &= \frac{1}{4\pi}\int_{-\infty}^{\infty} \ln\left(1 + \frac{\beta_{xy}^2/m_{xy}}{1+\omega^2/\alpha_y^2}\right)d\omega\\
&= \frac{\alpha_y}{2}\left(\sqrt{1+\beta_{xy}^2/m_{xy}} - 1\right).
\end{aligned} \tag{C9}$$

Dividing right-hand side of the latter equation by $\max\{\alpha_x, \alpha_y\}$, one gets the reduced infinite-history TE $T^{\infty}_{Y\to X}$ (10).

APPENDIX D: DETAILED RESULTS

Throughout this Appendix, let $m_{xy} > 1$ for definiteness, i.e., Y is a fast subsystem, and X is a slow one.

1. Unidirectional couplings

This is a limit case of essential coupling difference with $L = \infty$ and $\beta^2 = 0$, where $l_{xy}\beta^2 = \beta_{xy}^2 > 0$ and $l_{yx}\beta^2 = 0$ for the coupling $Y \to X$. It holds $\tilde{\Delta} = 1$ and $S_{Y\to X} = N_{Y\to X}$. All coupling quantifiers $X \to Y$ are zero.

To relate first the long-term DCE to the immediate and relative coupling strengths, consider a unidirectional coupling from the fast source $Y \to X$. Then the long-term DCE equals the relative coupling strength $S_{Y\to X} \approx \beta_{xy}^2/M$. A moderately strong coupling $1 \ll \beta_{xy}^2 \ll M$ means that the variance of the coupling term $k_{xy}y$ in the right-hand side of the evolution equation of the system is much greater than that of the individual term $-\alpha_x x$. Such coupling provides a small long-term DCE $S_{Y\to X} \ll 1$. To cause a large long-term DCE, the variance of the coupling (fast) term should exceed the variance of the individual (slow) term more than M times. If their ratio equals M, so the immediate coupling strength is large, then the long-term DCE is just equal to 1 r.u., i.e., this immediately strong coupling leads to an increase of the recipient variance only by 1 r.u. (doubling of the variance). The factor M can be understood as follows. The variance of the integral of the fast term over any time interval $\tau \gg \tau_{\text{char}} = 1/\alpha_y$ rises linearly with τ and equals the variance of this term multiplied by τ/α_y, since its ACF decays over the interval $1/\alpha_y$. Due to inertia of the slow recipient X, contribution of each term to the recipient variance is determined by the variance of the integral of that term over the interval $\tau = 1/\alpha_x \gg 1/\alpha_y$. For the slow term, this variance of integral is just the ordinary variance multiplied by $1/\alpha_x^2$. Therefore, the ratio of the contribution of the fast term to that of the slow term equals the ratio of their variances β_{xy}^2 divided by M.

For the unidirectional coupling from the slow source $X \to Y$, the long-term DCE equals the immediate coupling strength $S_{X\to Y} \approx \beta_{yx}^2$, being much smaller than the relative coupling strength $M\beta_{yx}^2$. Why is the long-term DCE not large for moderately weak coupling $1/M \ll \beta_{yx}^2 \ll 1$ due to possibly greater contribution of the slow (coupling) term than that of the fast (individual) term? Roughly speaking, this is because the contributions should be computed over the small relaxation time $1/\alpha_y$ of the fast recipient Y rather than over $1/\alpha_x$. Moreover, the individual term remains fast only if the long-term DCE of the slow coupling term is not large. So the ratio of these contributions equals the ratio of the variances β_{yx}^2 contrary to the case of the slow recipient X. Thus, the long-term DCE equals either the immediate (from the slow source) or the relative (from the fast source) coupling strength.

The original TE from the fast source is $T_{Y\to X} = \beta_{xy}^2/(4m_{xy})$ for any coupling strength β_{xy}^2, since cross correlation is always small $r^2 \ll 1$ for this unidirectional coupling (Table I). Hence, the long-term DCE is expressed via the original TE simply as $S_{Y\to X} = 4T_{Y\to X}$. The infinite-history TE reads $T^{\infty}_{Y\to X} = T_{Y\to X}$ for $\beta_{xy}^2/M \ll 1$ and $T^{\infty}_{Y\to X} = \sqrt{\beta_{xy}^2/(4M)} = \sqrt{T_{Y\to X}}$ for $\beta_{xy}^2/M \gg 1$. The long-term DCE and both TEs are small for relatively weak coupling $\beta_{xy}^2/M \ll 1$ and large for relatively strong coupling $\beta_{xy}^2/M \gg 1$.

The original TE from the slow source $T_{X\to Y} = \beta_{yx}^2(1 - r^2)/4$ simplifies to $T_{X\to Y} = \beta_{yx}^2/4 \ll 1$ for weak coupling $\beta_{yx}^2 \ll 1$ since the cross correlation is then small $r^2 = \beta_{yx}^2$ (Table I). The infinite-history TE is equal to $T^{\infty}_{X\to Y} = T_{X\to Y}$ for essentially weak coupling ($\beta_{yx}^2 \ll m_{yx} = 1/M$) and to $T^{\infty}_{X\to Y} = \sqrt{\beta_{yx}^2/(4M)} \ll T_{X\to Y}$ for moderately weak coupling ($1/M \ll \beta_{yx}^2 \ll 1$). The infinite-history TE changes its relationship to other quantifiers, depending on the relative coupling strength β_{yx}^2/m_{yx}. In total, for the weak coupling from the slow source, the "four-TE" relationship $S_{X\to Y} = 4T_{X\to Y}$ holds again, with $S_{X\to Y} = 4T^{\infty}_{X\to Y}$ for essentially weak coupling (where $T^{\infty}_{X\to Y} \ll 1/M$) and $S_{X\to Y} = 4M(T^{\infty})^2_{X\to Y} \ll 1$ for moderately weak coupling (with $1/M \ll T^{\infty}_{X\to Y} \ll 1/\sqrt{M}$). These relationships hold for small values of the coupling quantifiers which can be diagnosed in practice.

For strong coupling from the slow source $\beta_{yx}^2 \gg 1$, the value of r^2 gets close to unity (Table I). Then for moderately strong coupling, $1 \ll \beta_{yx}^2 \ll M$, one has $r^2 = 1 - 1/\beta_{yx}^2$ and the long-term DCE $S_{X\to Y} = 1/(1 - r^2) \gg 1$. In this range of coupling strengths, the original TE is constant $T_{X\to Y} = 1/4$ nats, since the rising coupling strength β_{yx}^2 exactly compensates for the decreasing factor $1 - r^2$ on the right-hand side of Eq. (11). So $T_{X\to Y}$ is not informative in respect of the long-term DCE. The infinite-history TE remains small $1/\sqrt{M} \ll T^{\infty}_{X\to Y} \ll 1$ and the long-term DCE reads

TABLE I. Coupling quantifiers for unidirectional couplings versus the coupling strength β_{xy}^2 or β_{yx}^2. The rate difference is strong $m_{xy} = M \gg 1$. The sign "−||−" means "the same as in the above box."

β_{xy}^2 or β_{yx}^2	r^2	$S_{Y\to X}$	$N_{Y\to X}$	$4T_{Y\to X}$	$4T^{\infty}_{Y\to X}$	$S_{X\to Y}$	$N_{X\to Y}$	$4T_{X\to Y}$	$4T^{\infty}_{X\to Y}$
$\beta_{xy}^2 \ll M$	$\frac{\beta_{xy}^2}{M^2} \ll 1$	$\frac{\beta_{xy}^2}{M} \ll 1$	$\frac{\beta_{xy}^2}{M} \ll 1$	$\frac{\beta_{xy}^2}{M} \ll 1$	$\frac{\beta_{xy}^2}{M} \ll 1$	0	0	0	0
$\beta_{xy}^2 \gg M$	$\frac{1}{M} \ll 1$	$\frac{\beta_{xy}^2}{M} \gg 1$	$\frac{\beta_{xy}^2}{M} \gg 1$	$\frac{\beta_{xy}^2}{M} \gg 1$	$\sqrt{\frac{4\beta_{xy}^2}{M}} \gg 1$	0	0	0	0
$\beta_{yx}^2 \ll \frac{1}{M}$	$\beta_{yx}^2 \ll 1$	0	0	0	0	$\beta_{yx}^2 \ll 1$	$\beta_{yx}^2 \ll 1$	$\beta_{yx}^2 \ll 1$	$\beta_{yx}^2 \ll 1$
$\frac{1}{M} \ll \beta_{yx}^2 \ll 1$	$\beta_{yx}^2 \ll 1$	0	0	0	0	−‖−	−‖−	−‖−	$\sqrt{\frac{4\beta_{yx}^2}{M}} \ll 1$
$1 \ll \beta_{yx}^2 \ll M$	$1 - \frac{1}{\beta_{yx}^2}$	0	0	0	0	$\beta_{yx}^2 \gg 1$	$\beta_{yx}^2 \gg 1$	1	−‖−
$\beta_{yx}^2 \gg M$	$1 - \frac{1}{M}$	0	0	0	0	−‖−	−‖−	$\frac{\beta_{yx}^2}{M} \gg 1$	$\sqrt{\frac{4\beta_{yx}^2}{M}} \gg 1$

$S_{X\to Y} = 4M(T^{\infty})^2_{X\to Y}$. So one has $T^{\infty}_{X\to Y} \ll T_{X\to Y} \ll S_{X\to Y}$. An empirical sign of this situation is a small value of $T^{\infty}_{X\to Y}$ accompanied by the intermediate $T_{X\to Y} = 1/4$ nats. Finally, for essentially strong coupling $\beta^2_{yx} \gg M$, one gets constant $r^2 = 1 - 1/M$ and large values of all coupling quantifiers $T^{\infty}_{X\to Y} = \sqrt{\beta^2_{yx}/4M} \gg 1$, $T_{X\to Y} = \beta^2_{yx}/(4M)$, and $S_{X\to Y} = 4MT_{X\to Y} = 4M(T^{\infty})^2_{X\to Y}$. Thus, the strong coupling from the slow source leads to large cross-correlation and diverse relationships between the long-term DCE and both TEs. Notice a simple approximate relationship $S_{X\to Y} = r^2/(1-r^2)$ which is valid for weak and moderately strong coupling from the slow source (i.e., if $1 - r^2 \gg 1/M$) and can be used in practical estimation.

For bidirectional couplings below, introduce a separate notation for a unidirectional DCE $S^{\text{uni}}_{Y\to X}$ which is $S_{Y\to X}$ under an additional restriction of no coupling $X \to Y$. It reads $S^{\text{uni}}_{Y\to X} = \frac{l_{xy}\beta^2}{1+m_{xy}} = \frac{\beta^2_{xy}}{1+m_{xy}}$.

2. Weak bidirectional couplings

At positive feedback, consider so weak bidirectional coupling that all coupling quantifiers in both directions are small. In this case, the cross correlation is necessarily small $r^2 \ll 1$ and the mean coupling is weak $\beta^2 \ll 1$ corresponding to the relative system determinant $\tilde{\Delta}$ close to unity. All these conditions are met, if couplings in both directions are relatively weak: $\beta^2_{xy} \ll M$ and $\beta^2_{yx} \ll 1/M$. The resulting relationships appear quite close to the unidirectional case, apart from difference for the coupling-on DCE of the relatively deficient coupling. Namely, for any coupling direction it holds $N_{[\cdot]} = S^{\text{uni}}_{[\cdot]} = 4T_{[\cdot]} = 4T^{\infty}_{[\cdot]}$, where $[\cdot]$ stands for the coupling direction. For the relatively predominant coupling, it holds $S_{[\cdot]} = S^{\text{uni}}_{[\cdot]}$, while for the relatively deficient coupling this relationship changes to $S_{[\cdot]} = L_r S^{\text{uni}}_{[\cdot]}$ (if this is a moderately predominant coupling from the fast source or essentially deficient coupling from the slow source) or $S_{[\cdot]} = (LM)S^{\text{uni}}_{[\cdot]}$ (if this is a deficient coupling from the fast source), and anyway involves a large factor L_r or LM. This difference is explained by considerable contribution of the feedback loop to the coupling-on DCE of a relatively deficient coupling. Note that for each coupling direction the original and infinite-history TEs coincide, both being small, that gives an empirical sign of this situation in the form $T_{Y\to X} = T^{\infty}_{Y\to X} \ll 1$ and $T_{X\to Y} = T^{\infty}_{X\to Y} \ll 1/M$.

Relax the condition of relative weakness of the coupling $X \to Y$ and let it be only moderately weak $1/M \ll \beta^2_{yx} \ll 1$ maintaining weak mean coupling $\beta^2 \ll 1$. All the coupling quantifiers remain small and related in the same way (Table II) with the only change concerning the infinite-history TE from the slow source $T^{\infty}_{X\to Y} = \sqrt{\beta^2_{yx}/4M} \ll T_{X\to Y}$. An empirical sign of this situation includes $T^{\infty}_{X\to Y} \ll T_{X\to Y} \ll 1$, $1/M \ll T^{\infty}_{X\to Y} \ll 1/\sqrt{M}$, $T_{Y\to X} = T^{\infty}_{Y\to X} \ll 1$, and a special check for weakness of the mean coupling, e.g., based on Table II.

Relax the condition of the relative weakness of the coupling from the fast source maintaining weak mean coupling $\beta^2 \ll 1$. It implies essentially strong coupling from the fast source $\beta^2_{xy} \gg M$ and essentially weak coupling from the slow source $\beta^2_{yx} \ll 1/M$, so $l_{xy} = L \gg M$. The quantifiers of the coupling from the fast source are then all large and related as in the case of unidirectional coupling from the fast source $S_{Y\to X} = N_{Y\to X} = S^{\text{uni}}_{Y\to X} = 4T_{Y\to X} = 4(T^{\infty})^2_{Y\to X} = L\beta^2/M \gg 1$. The cross correlation and the opposite long-term DCEs remain small: the noise-on and unidirectional DCEs equal four times the infinite-history TE $N_{X\to Y} = S^{\text{uni}}_{X\to Y} = 4T^{\infty}_{X\to Y} = \beta^2/L \ll 1$, the coupling-on DCE is much greater $S_{X\to Y} = (L/M)S^{\text{uni}}_{X\to Y} = \beta^2/M \ll 1$, and the original TE takes position between them as $N_{X\to Y} \ll T_{X\to Y} = \beta^4/(4M) \ll S_{X\to Y}$. An empirical sign of this situation includes $1 \ll T^{\infty}_{Y\to X} \ll T_{Y\to X}$, $T^{\infty}_{X\to Y} \ll T^{\infty}_{X\to Y} \ll 1$, and a special check for weak mean coupling. Thus, the original TE starts to "move" to the coupling-on DCE, away from the infinite-history TE which remains "linked" to the noise-on DCE, as manifested more clearly in the situations of Sec. IV C.

3. Large cross correlation

Large cross correlation $1 - r^2 \ll 1$ is a specific situation, observed either if mean coupling strength is close to unity with positive feedback or coupling from the slow source is strong. It is most difficult for estimation of directional coupling quantifiers from time series of x and y since both variables almost coincide, so information about the couplings should be extracted from their quite weak deviations from each other. Even a moderate measurement noise can render the task unsolvable. Still, for very weak measurement noise, an observation of practical relevance is that an approximate relationship for the coupling from the slow source $S_{X\to Y} \approx 1/(1-r^2)$ is valid (Table II), except for essentially strong couplings from the slow source. Namely, a sufficient condition is $\beta^2_{yx} \ll M$. Violation of the condition $\beta^2_{yx} \ll M$ can be diagnosed if the original TE is large in one direction $T_{X\to Y} \gg 1$ (from the slow source), and small in another direction $T_{Y\to X} \ll 1$. All other cases with $1 - r^2 \ll 1$ are "marked" with different relationships (Table II).

This relationship for a coupling from the slow source can be generalized to involve also small ($r^2 \ll 1$) cross correlations. Then it takes the form $S_{X\to Y} = r^2/(1-r^2)$ (Table II). For a large cross correlation, it applies no matter whether the coupling $X \to Y$ is relatively predominant or not. For a small cross correlation, it applies only in case of relatively predominant coupling $X \to Y$. It applies also in case of negative feedback under the same conditions on the coupling strength. Anyway, it does not apply to the coupling-on DCE from the fast source.

4. Ratios of coupling quantifiers

To summarize the relationships between different coupling quantifiers in the same direction for the positive feedback, note that it always holds $S_{[\cdot]} \geqslant N_{[\cdot]} \geqslant S^{\text{uni}}_{[\cdot]} \geqslant 4T^{\infty}_{[\cdot]}$ and $S_{[\cdot]} \geqslant 4T_{[\cdot]} \geqslant 4T^{\infty}_{[\cdot]}$. Approximate equalities take place for weak-enough couplings except for the coupling-on DCE of relatively deficient coupling. The value of $4T_{[\cdot]}$ can either exceed the respective $N_{[\cdot]}$ and $S^{\text{uni}}_{[\cdot]}$ or be less than these DCEs.

Concerning ratios of the quantifiers of the same kind in the opposite directions, note first the ratio of the coupling-on DCEs which always exactly equals $S_{Y\to X}/S_{X\to Y} = l_{xy}s$. The DCE for a relatively predominant coupling is always

TABLE II. Coupling quantifiers for bidirectional couplings with $m_{xy} = M \gg 1$. The first column gives the feedback sign s in the symbolic form. The third column gives the values of the coupling strength, where the parentheses are used for compactness to denote inequalities in strong sense, e.g., $(\frac{M}{L}, 1)$ means $\frac{M}{L} \ll \beta^2 \ll 1$. If β^2 is close to unity for positive feedback, then the value of the system determinant $\tilde{\Delta} = 1 - \beta^2$ is given instead. The asterisk in the third column means that the case of $L/M < M$ is presented there, while the opposite case is analogous.

s	l_{xy}	β^2	r^2	$S_{Y\to X}$	$N_{Y\to X}$	$4T_{Y\to X}$	$4T^{\infty}_{Y\to X}$	$S_{X\to Y}$	$N_{X\to Y}$	$4T_{X\to Y}$	$4T^{\infty}_{X\to Y}$
$\pm$	$\gg M$	$\ll M/L$	$\frac{L\beta^2}{M^2} \ll 1$	$\frac{L\beta^2}{M} \ll 1$	$\frac{L\beta^2}{M} \ll 1$	$\frac{L\beta^2}{M} \ll 1$	$\frac{L\beta^2}{M} \ll 1$	$\pm\frac{\beta^2}{M} \ll 1$	$\frac{\beta^2}{L} \ll 1$	$\frac{\beta^2}{L} \ll 1$	$\frac{\beta^2}{L} \ll 1$
		$(\frac{M}{L}, 1)$	$\frac{1}{M} \ll 1$	$\frac{L\beta^2}{M} \gg 1$	$\frac{L\beta^2}{M} \gg 1$	$\frac{L\beta^2}{M} \gg 1$	$\sqrt{\frac{4L\beta^2}{M}} \gg 1$	$-\|-$	$-\|-$	$\frac{\beta^4}{M} \ll 1$	$-\|-$
$+$	$\gg M$	$\tilde{\Delta} \in (\frac{1}{M}, 1)$	$\frac{1}{M\tilde{\Delta}} \ll 1$	$\frac{L}{M\tilde{\Delta}} \gg 1$	$\frac{L}{M} \gg 1$	$\frac{L}{M} \gg 1$	$\sqrt{\frac{4L}{M}} \gg 1$	$\frac{1}{M\tilde{\Delta}} \ll 1$	$\frac{1}{L\tilde{\Delta}} \ll 1$	$\frac{1}{M\tilde{\Delta}} \ll 1$	$\frac{1}{L} \ll 1$
		$\tilde{\Delta} \ll \frac{1}{M}$	$1 - M\tilde{\Delta}$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\frac{1}{M\tilde{\Delta}} \gg 1$	$\frac{M}{L} \ll 1$	1	$-\|-$
$-$	$\gg M$	$(1, \frac{L}{M})^*$	$\frac{1}{M\beta^2} \ll 1$	$\frac{L}{M} \gg 1$	$\frac{L\beta^2}{M} \gg 1$	$\frac{L\beta^2}{M} \gg 1$	$\sqrt{\frac{4L\beta^2}{M}} \gg 1$	$-\frac{1}{M} \ll 1$	$\frac{1}{L} \ll 1$	$\frac{\beta^2}{M} \ll 1$	$\frac{\beta^2}{L} \ll 1$
		$(\frac{L}{M}, M)$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\sqrt{\frac{4\beta^2}{LM}} \ll 1$
		$\gg M$	$-\|-$	$-\|-$	$L \gg 1$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\frac{\beta^2}{M} \gg 1$	$\sqrt{\frac{4\beta^2}{LM}}$
$\pm$	$(1, M)$	$\ll \frac{L}{M}$	$\frac{\beta^2}{L} \ll 1$	$\pm\beta^2 \ll 1$	$\frac{L\beta^2}{M} \ll 1$	$\frac{L\beta^2}{M} \ll 1$	$\frac{L\beta^2}{M} \ll 1$	$\frac{\beta^2}{L} \ll 1$	$\frac{\beta^2}{L} \ll 1$	$\frac{\beta^2}{L} \ll 1$	$\frac{\beta^2}{L} \ll 1$
		$(\frac{L}{M}, 1)$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\sqrt{\frac{\beta^2}{LM}} \ll 1$
$+$	$(1, M)$	$\tilde{\Delta} \in (\frac{1}{L}, 1)$	$\frac{1}{L\tilde{\Delta}} \ll 1$	$\frac{1}{\tilde{\Delta}} \gg 1$	$\frac{L}{M} \ll 1$	$\frac{L}{M} \ll 1$	$\frac{L}{M} \ll 1$	$\frac{1}{L\tilde{\Delta}} \ll 1$	$\frac{1}{L\tilde{\Delta}} \ll 1$	$\frac{1}{L\tilde{\Delta}} \ll 1$	$\sqrt{\frac{4}{LM}} \ll 1$
		$\tilde{\Delta} \in (\frac{1}{M}, \frac{1}{L})$	$1 - L\tilde{\Delta}$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\frac{1}{L\tilde{\Delta}} \gg 1$	$\frac{1}{L\tilde{\Delta}} \gg 1$	1	$-\|-$
		$\tilde{\Delta} \ll \frac{1}{M}$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\frac{M}{L} \gg 1$	$-\|-$	$-\|-$
$-$	$(1, M)$	$(1, \frac{M}{L})$	$\frac{1}{L} \ll 1$	$-1 + \frac{1}{\beta^2}$	$\frac{L\beta^2}{M} \ll 1$	$\frac{L\beta^2}{M} \ll 1$	$\frac{L\beta^2}{M} \ll 1$	$\frac{1}{L} \ll 1$	$\frac{1}{L} \ll 1$	$\frac{1}{L} \ll 1$	$\sqrt{\frac{4\beta^2}{LM}} \ll 1$
		$(\frac{M}{L}, M)$	$\frac{M}{L\beta^2} \ll 1$	$-1 + \frac{L}{M}$	$\frac{L\beta^2}{M} \gg 1$	$\frac{L\beta^2}{M} \gg 1$	$\sqrt{\frac{4L\beta^2}{M}} \gg 1$	$-\|-$	$-\|-$	$\frac{\beta^2}{M} \ll 1$	$-\|-$
		$\gg M$	$-\|-$	$-\|-$	$L \gg 1$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\frac{\beta^2}{M} \gg 1$	$\sqrt{\frac{\beta^2}{LM}}$
$\pm$	$(\frac{1}{M}, 1)$	$\ll \frac{1}{LM}$	$L\beta^2 \ll 1$	$\pm\beta^2 \ll 1$	$\frac{\beta^2}{LM} \ll 1$	$\frac{\beta^2}{LM} \ll 1$	$\frac{\beta^2}{LM} \ll 1$	$L\beta^2 \ll 1$	$L\beta^2 \ll 1$	$L\beta^2 \ll 1$	$L\beta^2 \ll 1$
		$(\frac{1}{LM}, \frac{1}{L})$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\sqrt{\frac{4L\beta^2}{M}} \ll 1$
		$(\frac{1}{L}, 1)$	$1 - \frac{1}{L\beta^2}$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$L\beta^2 \gg 1$	$L\beta^2 \gg 1$	1	$-\|-$
$+$	$(\frac{1}{M}, 1)$	$\tilde{\Delta} \in (\frac{1}{M}, 1)$	$1 - \frac{\tilde{\Delta}}{L}$	$\frac{1}{\tilde{\Delta}} \gg 1$	$\frac{1}{LM} \ll 1$	$\frac{1}{LM} \ll 1$	$\frac{1}{LM} \ll 1$	$\frac{L}{\tilde{\Delta}} \gg 1$	$\frac{L}{\tilde{\Delta}} \gg 1$	1	$\sqrt{\frac{4L}{M}} \ll 1$
		$\tilde{\Delta} \ll \frac{1}{M}$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$LM \gg 1$	$-\|-$	$-\|-$
$-$	$(\frac{1}{M}, 1)$	$(1, \frac{M}{L})$	$1 - \frac{1}{L}$	$-1 + \frac{1}{\beta^2}$	$\frac{\beta^2}{LM} \ll 1$	$\frac{\beta^2}{LM} \ll 1$	$\frac{\beta^2}{LM} \ll 1$	$L \gg 1$	$L \gg 1$	1	$\sqrt{\frac{4L\beta^2}{M}} \ll 1$
		$(\frac{M}{L}, M)$	$1 - \frac{\beta^2}{M}$	$-\|-$	$-\|-$	$\frac{\beta^4}{M^2} \ll 1$	$-\|-$	$-\|-$	$-\|-$	$\frac{L\beta^2}{M} \gg 1$	$\sqrt{\frac{4L\beta^2}{M}} \gg 1$
		(M, LM)	$\frac{M}{\beta^2} \ll 1$	$-1 + \frac{1}{M}$	$\frac{1}{L} \ll 1$	$\frac{\beta^2}{M} \gg 1$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$
		$\gg LM$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\sqrt{\frac{4\beta^2}{LM}} \gg 1$	$-\|-$	$-\|-$	$-\|-$	$-\|-$
$\pm$	$\ll \frac{1}{M}$	$\ll \frac{1}{LM}$	$L\beta^2 \ll 1$	$\pm\beta^2 \ll 1$	$\frac{\beta^2}{LM} \ll 1$	$\frac{\beta^2}{LM} \ll 1$	$\frac{\beta^2}{LM} \ll 1$	$L\beta^2 \ll 1$	$L\beta^2 \ll 1$	$L\beta^2 \ll 1$	$L\beta^2 \ll 1$
		$(\frac{1}{LM}, \frac{1}{L})$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\sqrt{\frac{4L\beta^2}{M}} \ll 1$
		$(\frac{1}{L}, \frac{M}{L})$	$1 - \frac{1}{L\beta^2}$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$L\beta^2 \gg 1$	$L\beta^2 \gg 1$	1	$-\|-$
		$(\frac{M}{L}, 1)$	$1 - \frac{1}{M}$	$-\|-$	$-\|-$	$\frac{\beta^4}{M^2} \ll 1$	$-\|-$	$-\|-$	$-\|-$	$\frac{L\beta^2}{M} \gg 1$	$\sqrt{\frac{4L\beta^2}{M}} \gg 1$
$+$	$\ll \frac{1}{M}$	$\tilde{\Delta} \in (\frac{1}{M}, 1)$	$1 - \frac{\tilde{\Delta}}{M}$	$\frac{1}{\tilde{\Delta}} \gg 1$	$\frac{1}{LM} \ll 1$	$\frac{1}{M^2} \ll 1$	$\frac{1}{LM} \ll 1$	$\frac{L}{\tilde{\Delta}} \gg 1$	$\frac{L}{\tilde{\Delta}} \gg 1$	$\frac{L}{M} \gg 1$	$\sqrt{\frac{4L}{M}} \gg 1$
		$\tilde{\Delta} \ll \frac{1}{M}$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$LM \gg 1$	$-\|-$	$-\|-$
$-$	$\ll \frac{1}{M}$	$(1, M)$	$1 - \frac{\beta^2}{M}$	$-1 + \frac{1}{\beta^2}$	$\frac{\beta^2}{LM} \ll 1$	$\frac{\beta^4}{M^2} \ll 1$	$\frac{\beta^2}{LM} \ll 1$	$L \gg 1$	$L \gg 1$	$\frac{L\beta^2}{M} \gg 1$	$\sqrt{\frac{4L\beta^2}{M}} \gg 1$
		(M, LM)	$\frac{M}{\beta^2} \ll 1$	$-1 + \frac{1}{M}$	$\frac{1}{L} \ll 1$	$\frac{\beta^2}{M} \gg 1$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$-\|-$
		$\gg LM$	$-\|-$	$-\|-$	$-\|-$	$-\|-$	$\sqrt{\frac{4\beta^2}{LM}} \gg 1$	$-\|-$	$-\|-$	$-\|-$	$-\|-$

positive. An interesting situation is a moderately predominant (relatively deficient) coupling from the fast source $Y \to X$ for negative feedback, where the CCF plot is unusual [dashed line in Fig. 2(b)]. Then the respective coupling-on DCE is negative $S_{Y\to X} < 0$, but in absolute value it is l_{xy} times as large as the opposite DCE $S_{X\to Y} > 0$. Thus, a decrease of the recipient variance due to switching the coupling on is possible even for a (moderately) predominant coupling.

Assume positive feedback below, until the subsection "Negative feedback." The ratio of the unidirectional DCEs is exactly equal to $S^{\mathrm{uni}}_{Y\to X}/S^{\mathrm{uni}}_{X\to Y} = l^2_{xy}/m_{xy}$ for any parameters. If coupling from the slow source is predominant, then its unidirectional DCE prevails by the factor of L^2M. If coupling from the fast source is essentially predominant, then its unidirectional DCE prevails by the factor of L^2/M. In case of moderately predominant coupling from the fast source $l_{xy} \gg 1$, its unidirectional DCE L^2/M can be greater or less than unity depending on the values of L and M. Thus, take as a vivid numerical example $M = 100$ and $l_{xy} = \sqrt{50} \approx 7$, e.g., $\beta^2_{xy} = 5$ and $\beta^2_{yx} = 0.1$ where the mean coupling strength $\beta^2 = \sqrt{0.5} \approx 0.7$ is intermediate. Then, the unidirectional DCE from the fast source $S^{\mathrm{uni}}_{Y\to X} \approx 0.05$ r.u. is twice as small as that from the slow source $S^{\mathrm{uni}}_{X\to Y} \approx 0.1$ r.u., both being small. The coupling-on DCE from the fast source is moderately large $S_{Y\to X} \approx 2.5$ r.u. (the slow recipient variance increases 3.5 times due to switching the coupling $Y \to X$ on). This effect is 7 times as large as $S_{Y\to X} \approx 0.4$ r.u. which is moderately small (the fast recipient variance rises only 1.4 times due to switching the coupling $X \to Y$ on). So this predominant coupling from the fast source appears to be also predominant in terms of the coupling-on DCE, but deficient in terms of the unidirectional DCE.

The ratio of the noise-on DCEs $N_{Y\to X}/N_{X\to Y}$ for the predominant coupling from the fast source ranges from L^2/M for weak-enough mean coupling to smaller value of L^2/M^2 for the mean coupling strength close to unity (Table II). Therefore, for moderately predominant coupling from the fast source ($L \ll M$), this ratio can be greater than unity for weak-enough mean coupling and inevitably gets less than unity for small enough $\tilde{\Delta}$. Such a transition does not take place for the coupling-on and unidirectional DCEs. For the deficient coupling from the fast source, this ratio ranges respectively from $1/(L^2M)$ to $1/(L^2M^2)$, both values being small. Thus, predominant coupling from the slow source prevails even stronger in terms of its noise-on DCE.

The ratio of the original TEs $T_{Y\to X}/T_{X\to Y}$ for the predominant coupling from the fast source ranges from L^2/M for weak-enough mean coupling to smaller value of L/M for the mean coupling strength close to unity (Table II). The latter is inevitably small for the moderately predominant coupling $Y \to X$, while the former can exceed unity similarly to the ratio of noise-on DCEs. For the deficient coupling from the fast source, the ratio $T_{Y\to X}/T_{X\to Y}$ ranges respectively from $1/(L^2M)$ to $1/(LM)$, both values being small.

The ratio of the infinite-history TEs $T^{\infty}_{Y\to X}/T^{\infty}_{X\to Y}$ for the predominant coupling from the fast source ranges from L^2/M for weak-enough mean coupling to $\sqrt{4L^3/M}$ for the mean coupling strength close to unity. In the latter case, this ratio can be greater than unity when the ratio of the original TEs is less than unity, so the moderately predominant coupling $Y \to X$ can be predominant in terms of the infinite-history TE and deficient in terms of the original TE and noise-on DCE. For the deficient coupling from the fast source, the ratio $T^{\infty}_{Y\to X}/T^{\infty}_{X\to Y}$ ranges from $1/(L^2M)$ to $\sqrt{4/(L^3M)}$, i.e., always small.

5. Negative feedback

For $s = -1$ and weak mean coupling ($\beta^2 \ll 1$) almost everything remains the same as in case of positive feedback with the only difference that the coupling-on DCE in the direction of relatively deficient coupling is negative. Similarity of both cases is reflected by the sign $\pm$ in the leftmost column of Table II. What is quite different in case of negative feedback is that the mean coupling strength can be arbitrarily large maintaining stationarity of the process (X_t, Y_t). Thus, different intervals of large $\beta^2 \gg 1$ give rise to different scaling regimes of the coupling quantifiers under study.

Both ratios $T_{Y\to X}/T_{X\to Y}$ and $T^{\infty}_{Y\to X}/T^{\infty}_{X\to Y}$ tend to l_{xy} if β^2 is large enough: $\beta^2 \gg LM$ and $\beta^2 \gg M$, respectively. So the quantifier in the direction of predominant coupling is L times as large as that in the opposite direction. For weaker mean couplings, these ratios are somewhat smaller for predominant coupling from the fast source and their inverses are even greater for the predominant coupling from the slow source. In the former case, both ratios can become less than unity for $L^2 < M$. The ratio of noise-on DCEs $N_{Y\to X}/N_{X\to Y}$ tends to l^2_{xy} for strong-enough mean coupling corresponding to essentially strong coupling from the fast source and strong coupling from the slow source. So the noise-on DCE is greater by the factor of L^2 in the direction of predominant coupling. The ratio of the unidirectional DCEs $S^{\mathrm{uni}}_{Y\to X}/S^{\mathrm{uni}}_{X\to Y}$ is l^2_{xy}/m_{xy} as for the positive feedback.

6. Numerics at boundaries

The above relationships remain reasonably accurate even at boundary points. Table III confirms this statement for the boundary points in terms of mean coupling strength β^2 for the essentially predominant coupling from the fast source. There, the values obtained from the exact Eqs. (7)–(10) are presented along with two numbers in parentheses which are the estimates obtained from the asymptotic expressions for the typical intervals divided by the boundary point. If both numbers in parentheses coincide with the actual boundary value, then they are omitted.

A particular boundary case of equal rates $M = 1$ is presented in Table IV, where the two-TE relationship $S_{Y\to X} = 2T_{Y\to X}$ should be used instead of the four-TE law. Still, this difference is not very large. Quite different situation occurs when coupling-on DCEs in both directions are zero for negative feedback and relatively equivalent couplings $l_{xy} = m_{xy}$ as clear also from Eq. (7). The reason is that the contributions of the two terms on the right-hand side are equal in absolute value but opposite in sign. This specific balance is not captured by both TEs which take the same large values as in case of positive feedback and large coupling-on DCEs.

TABLE III. Coupling quantifiers for boundary values of β^2 (β_{xy}^2) for essentially predominant (unidirectional) coupling $Y \to X$. Figure brackets give two approximate values obtained via asymptotic formulas (from Table II) for the two intervals of β^2 (β_{xy}^2) divided by the boundary point. The approximate values are omitted if they coincide with the actual boundary value.

s	l_{xy}	β^2	r^2	$S_{Y\to X}$	$N_{Y\to X}$	$4T_{Y\to X}$	$4T_{Y\to X}^{\infty}$	$S_{X\to Y}$	$N_{X\to Y}$	$4T_{X\to Y}$	$4T_{X\to Y}^{\infty}$
	∞	$\beta_{xy}^2 = M$	$\frac{1}{2M} \ll 1$	1	1	1	$2(\sqrt{2}-1)$	0	0	0	0
			$\{\frac{1}{M}, \frac{1}{M}\}$				$\{1, 2\}$				
$\pm$	$\gg M$	M/L	$\frac{1}{2M} \ll 1$	1	1	1	$2(\sqrt{2}-1)$	$\pm\frac{1}{L} \ll 1$	$\frac{M}{L^2} \ll 1$	$\frac{2M}{L^2} \ll 1$	$\frac{M}{L^2} \ll 1$
			$\{\frac{1}{M}, \frac{1}{M}\}$				$\{1, 2\}$			$\{\frac{M}{L^2}, \frac{M}{L^2}\}$	
$+$	$\gg M$	$\frac{1}{2}$	$\frac{1}{M} \ll 1$	$\frac{L}{M} \gg 1$	$\frac{L}{2M} \gg 1$	$\frac{L}{2M} \gg 1$	$\sqrt{\frac{2L}{M}} \gg 1$	$\frac{1}{M} \ll 1$	$\frac{1}{2L} \ll 1$	$\frac{1}{2M} \ll 1$	$\frac{1}{2L} \ll 1$
			$\{\frac{1}{M}, \frac{2}{M}\}$	$\{\frac{L}{2M}, \frac{2L}{M}\}$	$\{\frac{L}{2M}, \frac{L}{M}\}$	$\{\frac{L}{2M}, \frac{L}{M}\}$	$\{\sqrt{\frac{2L}{M}}, \sqrt{\frac{4L}{M}}\}$	$\{\frac{1}{2M}, \frac{2}{M}\}$	$\{\frac{1}{2L}, \frac{2}{L}\}$	$\{\frac{1}{4M}, \frac{2}{M}\}$	$\{\frac{1}{2L}, \frac{1}{L}\}$
		$\tilde{\Delta} = \frac{1}{M}$	$\frac{1}{M} \ll 1$	$L \gg 1$	$\frac{L}{M} \gg 1$	$\frac{2L}{M} \gg 1$	$\sqrt{\frac{4L}{M}} \gg 1$	1	$\frac{M}{2L} \ll 1$	1	$\frac{1}{L} \ll 1$
			$\{1, 0\}$			$\{\frac{L}{M}, \frac{L}{M}\}$			$\{\frac{M}{L}, \frac{M}{L}\}$		
$-$	$\gg M$	1	$\frac{1}{M} \ll 1$	$\frac{L}{2M} \gg 1$	$\frac{L}{M} \gg 1$	$\frac{L}{M} \gg 1$	$\sqrt{\frac{4L}{M}} \gg 1$	$-\frac{1}{2M}$	$\frac{2}{3L} \ll 1$	$\frac{1}{2M}$	$\frac{1}{L}$
				$\{\frac{L}{M}, \frac{L}{M}\}$				$-\{\frac{1}{M}, \frac{1}{M}\}$	$\{\frac{1}{L}, \frac{1}{L}\}$	$\{\frac{1}{M}, \frac{1}{M}\}$	
		$\frac{L}{M}^*$	$\frac{1}{L} \ll 1$	$\frac{L}{M} \gg 1$	$\frac{L^2}{M^2} \gg 1$	$\frac{L^2}{M^2} \gg 1$	$\frac{2L}{M} \gg 1$	$-\frac{1}{M}$	$\frac{1}{L} \ll 1$	$\frac{L}{M^2}$	$\frac{2(\sqrt{2}-1)}{M}$
											$\{\frac{1}{M}, \frac{2}{M}\}$
		M	$\frac{1}{M^2} \ll 1$	$L \gg 1$	$\frac{L}{2} \gg 1$	$L \gg 1$	$\sqrt{4L} \gg 1$	$-\frac{1}{M}$	$\frac{1}{L} \ll 1$	1	$\frac{4}{\sqrt{L}} \ll 1$
					$[L, L]$						

APPENDIX E: NUMERICS OF COUPLING QUANTIFIERS

Various coupling quantifiers expressed via dimensionless parameters for different typical intervals and boundary values of the mean coupling strength β^2 are summarized in Tables I–IV.

The relationships between the coupling quantifiers and the dimensionless parameters are found as intermediate asymptotic in the approximate power-law form. Namely, one takes, e.g., $m_{xy} \gg 1$ and $l_{xy} \gg m_{xy}$ (or other strong inequalities) to expand Eqs. (7)–(10) into Taylor series with respect to small parameters $1/m_{xy}$ and m_{xy}/l_{xy} (or others) and retain the lowest order. As an example, consider essentially predominant coupling from the fast source $Y \to X$ (i.e., $l_{xy} \gg m_{xy} \gg 1$) for positive feedback $s = 1$ and weak-enough mean coupling $\beta^2 \ll M/L \ll 1$ (the first line of Table II). Then Eq. (7) in the

TABLE IV. Coupling quantifiers for bidirectional coupling and equal relaxation rates $m_{xy} = M = 1$. Notice the products $2T_{[\cdot]}$ instead of $4T_{[\cdot]}$.

s	l_{xy}	β^2	r^2	$S_{Y\to X}$	$N_{Y\to X}$	$2T_{Y\to X}$	$2T_{Y\to X}^{\infty}$	$S_{X\to Y}$	$N_{X\to Y}$	$2T_{X\to Y}$	$2T_{X\to Y}^{\infty}$
$\pm$	$\gg 1$	$\ll 1/L$	$\frac{L\beta^2}{4} \ll 1$	$\frac{L\beta^2}{2} \ll 1$	$\frac{L\beta^2}{2} \ll 1$	$\frac{L\beta^2}{2} \ll 1$	$\frac{L\beta^2}{2} \ll 1$	$\pm\frac{\beta^2}{2} \ll 1$	$\frac{\beta^2}{2L} \ll 1$	$\frac{\beta^2}{2L} \ll 1$	$\frac{\beta^2}{2L} \ll 1$
		$\frac{1}{L} \ll \cdot \ll 1$	$\frac{1}{2}$	$\frac{L\beta^2}{2} \gg 1$	$\frac{L\beta^2}{2} \gg 1$	$\frac{L\beta^2}{4} \gg 1$	$\sqrt{L\beta^2} \gg 1$	$-\Vert-$	$-\Vert-$	$\frac{\beta^4}{8} \ll 1$	$-\Vert-$
$+$	$\gg 1$	$\frac{2}{3}$	$\frac{3}{4}$	$L \gg 1$	$\frac{L}{2} \gg 1$	$\frac{L}{6} \gg 1$	$\sqrt{\frac{2L}{3}} \gg 1$	1	$\frac{1}{2L} \ll 1$	$\frac{1}{12}$	$\frac{1}{3L} \ll 1$
		$\tilde{\Delta} \ll 1$	$1 - \tilde{\Delta}$	$\frac{L}{2\tilde{\Delta}} \gg 1$	$L \gg 1$	$\frac{L}{4} \gg 1$	$\sqrt{L} \gg 1$	$\frac{1}{2\tilde{\Delta}} \gg 1$	$\frac{1}{L} \ll 1$	$\frac{1}{4}$	$\frac{1}{2L}$
$-$	$\gg 1$	$1 \ll \cdot \ll L$	$1 - \frac{1}{\beta^2}$	$\frac{L}{2} \gg 1$	$\frac{L}{2} \gg 1$	$\frac{L\beta^2}{4} \gg 1$	$\sqrt{L\beta^2} \gg 1$	$-\frac{1}{2}$	$\frac{1}{2L} \ll 1$	$\frac{\beta^2}{4} \gg 1$	$\frac{\beta^2}{2L} \ll 1$
		$\gg L$	$-\Vert-$	$-\Vert-$	$-\Vert-$	$-\Vert-$	$-\Vert-$	$-\Vert-$	$-\Vert-$	$-\Vert-$	$\sqrt{\frac{\beta^2}{L}} \gg 1$
$+$	1	$\ll 1$	$\beta^2 \ll 1$	$\beta^2 \ll 1$	$\frac{\beta^2}{2} \ll 1$	$\frac{\beta^2}{2} \ll 1$	$\frac{\beta^2}{2} \ll 1$	$\beta^2 \ll 1$	$\frac{\beta^2}{2} \ll 1$	$\frac{\beta^2}{2} \ll 1$	$\frac{\beta^2}{2} \ll 1$
		$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\sqrt{\frac{3}{2}} - 1$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\sqrt{\frac{3}{2}} - 1$
		$\tilde{\Delta} \ll 1$	$1 - \tilde{\Delta}$	$\frac{1}{\tilde{\Delta}} \gg 1$	1	$\frac{1}{2}$	$\sqrt{2} - 1$	$\frac{1}{\tilde{\Delta}} \gg 1$	1	$\frac{1}{2}$	$\sqrt{2} - 1$
$-$	1	$\ll 1$	0	0	$\frac{\beta^2}{2} \ll 1$	$\frac{\beta^2}{2} \ll 1$	$\frac{\beta^2}{2} \ll 1$	0	$\frac{\beta^2}{2} \ll 1$	$\frac{\beta^2}{2} \ll 1$	$\frac{\beta^2}{2} \ll 1$
		1	$-\Vert-$	$-\Vert-$	$\frac{1}{3}$	$\frac{1}{2}$	$\sqrt{2} - 1$	$-\Vert-$	$\frac{1}{3}$	$\frac{1}{2}$	$\sqrt{2} - 1$
		$\gg 1$	$-\Vert-$	$-\Vert-$	1	$\frac{\beta^2}{2} \gg 1$	$\sqrt{\beta^2} \gg 1$	$-\Vert-$	1	$\frac{\beta^2}{2} \gg 1$	$\sqrt{\beta^2} \gg 1$

main text gives

$$S_{Y\to X} = \frac{l_{xy}\beta^2(1+m_{xy}/l_{xy})}{m_{xy}(1-\beta^2)(1+1/m_{xy})} = \frac{l_{xy}\beta^2}{m_{xy}}(1+m_{xy}/l_{xy}+\beta^2-1/m_{xy}+\dots) \approx \frac{l_{xy}\beta^2}{m_{xy}}.$$

The squared cross correlation is obtained from Eq. (C4) as

$$r^2 \approx \frac{\frac{l_{xy}\beta^2}{m_{xy}l_{xy}}\left(1+\frac{2m_{xy}}{l_{xy}}+\frac{m_{xy}^2}{l_{xy}^2}\right)}{\left(\frac{m_{xy}}{l_{xy}}+\beta^2+\frac{1}{l_{xy}}\right)\left(1+\frac{1}{m_{xy}}-\beta^2\right)} \approx \frac{l_{xy}\beta^2}{m_{xy}^2} \ll \frac{1}{m_{xy}} \ll 1.$$

The opposite coupling-on DCE $S_{X\to Y}$ equals $S_{Y\to X}/l_{xy}$ and so

$$S_{X\to Y} \approx \frac{\beta^2}{m_{xy}} \ll 1.$$

Thus, the reduced original TE $T_{Y\to X}$ is derived from Eq. (9) of the main text as

$$T_{Y\to X} \approx \frac{l_{xy}\beta^2}{4m_{xy}}\left(1-\frac{l_{xy}\beta^2}{m_{xy}^2}\right)\left(1+\frac{\beta^2}{m_{xy}}\right) \approx \frac{l_{xy}\beta^2}{4m_{xy}} \ll 1.$$

So $S_{Y\to X} \approx 4T_{Y\to X}$, both being small, as one can see in the first line of Table II.

For other ratios of dimensionless parameters, the Taylor series expansion may appear more complicated requiring analysis of several terms to find out which is of the lowest order. However, the principle is the same as in the above example.

To get exact values of TEs and DCEs at boundary points, one just substitutes these boundary values of dimensionless parameters into Eqs. (7)–(10).

Note that the mean coupling strength $\beta^2 = |\beta_{xy}\beta_{yx}|$ (not so obvious parameter) turns out to be an important dimensionless parameter whose characteristic values separate domains with different relationships of coupling quantifiers.

APPENDIX F: APPLICATION ISSUES

After getting the estimates of ACFs, CCF, m_{xy}, original, and/or infinite-history TEs, it is necessary to decide which of the diverse relationships between the coupling quantifiers should be used. It can be done on the basis of the above empirical signs of applicability and Tables I–IV. Thus, the case of relatively weak couplings in both directions is considered in Sec. V.

Another characteristic situation is met when $T^{\infty}_{Y\to X} = T_{Y\to X} \ll 1$, $1/M \ll T^{\infty}_{X\to Y} \ll 1/\sqrt{M}$, $1/M \ll T_{X\to Y} \ll 1$, and $T^{\infty}_{X\to Y} \ll T_{X\to Y}$. Here, the mean coupling strength β^2 may appear small or close to unity that determines the relationships between the coupling quantifiers (Table II). The CCF plot depends on which coupling is predominant. Note that $C_{xy}(\tau)$ typically reaches its main extremum at $\tau > 0$ (i.e., the process y leads) if the coupling from the fast source Y is predominant and at $\tau < 0$ otherwise, which is violated if the coupling from the fast source Y is moderately predominant and the feedback is negative. Then according to the CCF extremum, the process x is leading [Figs. 2(b) and 2(d)]. In this case, ACFs can be used to determine whether the mean coupling strength β^2 is small or not: Long-term values of the ACF of the fast subsystem Y are determined by the slow decay rate α_x only for β^2 close to unity. In addition, β^2 can be estimated as $\beta^2 = \sqrt{32MT_{Y\to X}T_{X\to Y}}$ or $\beta^2 = 4MT^{\infty}_{X\to Y}\sqrt{T^{\infty}_{Y\to X}}$. If it turns out $\beta^2 \ll 1$, then one gets $l_{xy} = \sqrt{MT_{Y\to X}/T_{X\to Y}}$, otherwise $l_{xy} = \sqrt{(M/2)T_{Y\to X}/(\beta^2 T_{X\to Y})}$. If it holds $1 \ll l_{xy} \ll M$ as a result, then Table II shows that for any mean coupling strength, a moderately deficient coupling from the slow source corresponds to $S_{X\to Y} = N_{X\to Y} = 4T_{X\to Y} = 4(T^{\infty})^2_{X\to Y}$, while its unidirectional DCE is $S^{\text{uni}}_{X\to Y} = S_{X\to Y}$ for $\beta^2 \ll 1$ and $S^{\text{uni}}_{X\to Y} = \beta^2 S_{X\to Y} \gg S_{X\to Y}$ for $\beta^2 \gg 1$. For the respective moderately predominant coupling from the fast source $S^{\text{uni}}_{Y\to X} = N_{Y\to X} = 4T_{Y\to X} = 4T^{\infty}_{Y\to X}$ and the coupling-on DCE is $S_{Y\to X} = S^{\text{uni}}_{Y\to X}$ for $\beta^2 \ll 1$ and $S_{Y\to X} = -1 + 1/\beta^2$ for $\beta^2 \gg 1$.

Other intervals of coupling strengths and predominance parameters can be handled based on Tables I–IV and typical plots of the correlation functions in the same manner.

Additional prior information or a full set of the autoregressive model coefficients can be used for a direct estimation of the long-term DCEs based on their definitions (4) and (5).

[1] K. Hlavackova-Schindler, M. Palus, M. Vejmelka, and J. Bhattacharya, Phys. Rep. **441**, 1 (2007).

[2] T. Bossomaier, L. Barnett, M. Harre, and J. T. Lizier, *An Introduction to Transfer Entropy. Information Flow in Complex Systems* (Springer, Switzerland, 2016).

[3] T. Schreiber, Phys. Rev. Lett. **85**, 461 (2000).

[4] M. Palus, V. Komarek, Z. Hrncir, and K. Sterbova, Phys. Rev. E **63**, 046211 (2001).

[5] M. Prokopenko and J. T. Lizier, Sci. Rep. **4**, 5394 (2014).

[6] J. T. Lizier, M. Prokopenko, and A. Y. Zomaya, Phys. Rev. E **77**, 026110 (2008).

[7] *Directed Information Measures in Neuroscience*, edited by M. Wibral, R. Vicente, and J. T. Lizier (Springer-Verlag, Berlin, 2014).

[8] M. Staniek and K. Lehnertz, Phys. Rev. Lett. **100**, 158101 (2008).

[9] J. Runge, J. Heitzig, V. Petoukhov, and J. Kurths, Phys. Rev. Lett. **108**, 258701 (2012).

[10] S. Stramaglia, G. R. Wu, M. Pellicoro, and D. Marinazzo, Phys. Rev. E **86**, 066211 (2012).

[11] J. Sun and E. M. Bollt, Physica D **267**, 49 (2014).

[12] M. Palus, Phys. Rev. Lett. **112**, 078702 (2014).

[13] X. S. Liang and R. Kleeman, Phys. Rev. Lett. **95**, 244101 (2005).

[14] X. S. Liang, Phys. Rev. E **94**, 052201 (2016).

[15] H. Ashikaga and R. G. James, Chaos **28**, 075306 (2018).

[16] E. M. Bollt, Chaos **28**, 075309 (2018).

[17] D. Chionis, A. Dokhane, H. Ferroukhi, and A. Pautz, Chaos **29**, 043126 (2019).
[18] A. Attanasio, A. Pasini, and U. Triacca, Atmos. Clim. Sci. **3**, 515 (2013).
[19] A. Kaiser and T. Schreiber, Physica D **166**, 43 (2002).
[20] M. Palus and M. Vejmelka, Phys. Rev. E **75**, 056211 (2007).
[21] M. Vejmelka and M. Palus, Phys. Rev. E **77**, 026214 (2008).
[22] A. Bahraminasab, F. Ghasemi, A. Stefanovska, P. V. E. McClintock, and H. Kantz, Phys. Rev. Lett. **100**, 084101 (2008)
[23] I. Vlachos and D. Kugiumtzis, Phys. Rev. E **82**, 016207 (2010).
[24] D. Kugiumtzis, Phys. Rev. E **87**, 062918 (2013).
[25] L. Faes, G. Nollo, and A. Porta, Phys. Rev. E **83**, 051112 (2011).
[26] M. Wibral, B. Rahm, M. Rieder, M. Lindner, R. Vicente, and J. Kaiser, Prog. Biophys. Mol. Biol. **105**, 80 (2011).
[27] J. Runge, J. Heitzig, N. Marwan, and J. Kurths, Phys. Rev. E **86**, 061121 (2012).
[28] J. Hlinka, D. Hartman, M. Vejmelka, J. Runge, N. Marwan, J. Kurths, and M. Palus, Entropy **15**, 2023 (2013).
[29] J. Runge, V. Petoukhov, J. F. Donges, J. Hlinka, N. Jajcay, M. Vejmelka, D. Hartman, N. Marwan, M. Palus, and J. Kurths, Nat. Commun. **6**, 8502 (2015).
[30] J. M. Amigo, R. Monetti, B. Graff, and G. Graff, Chaos **26**, 113115 (2016).
[31] J. Runge, P. Nowack, M. Kretschmer, S. Flaxman, and D. Sejdinovic, Sci. Adv. **5**, eaau4996 (2019).
[32] P.-O. Amblard and O. J. J. Michel, Entropy **15**, 113 (2013).
[33] N. Ay and D. Polani, Adv. Complex Syst. **11**, 17 (2008).
[34] J. T. Lizier and M. Prokopenko, Eur. Phys. J. B **73**, 605 (2010).
[35] D. W. Hahs and S. D. Pethel, Phys. Rev. Lett. **107**, 128701 (2011).
[36] D. A. Smirnov, Phys. Rev. E **87**, 042917 (2013).
[37] L. Faes, D. Kugiumtzis, G. Nollo, F. Jurysta, and D. Marinazzo, Phys. Rev. E **91**, 032904 (2015).
[38] R. G. James, N. Barnett, and J. P. Crutchfield, Phys. Rev. Lett. **116**, 238701 (2016).
[39] L. Faes, D. Marinazzo, and S. Stramaglia, Entropy **19**, 408 (2017).
[40] D. Marinazzo, L. Angelini, M. Pellicoro, and S. Stramaglia, Phys. Rev. E **99**, 040101(R) (2019).
[41] Ch. Koutlis, V. K. Kimiskidis, and D. Kugiumtzis, Int. J. Neural Syst. **29**, 1850051 (2019).
[42] P. A. Stokes and P. L. Purdon, Proc. Natl. Acad. Sci. USA **114**, E7063 (2017).
[43] D. A. Smirnov, Phys. Rev. E **90**, 062921 (2014).
[44] D. A. Smirnov and I. I. Mokhov, Phys. Rev. E **92**, 042138 (2015).
[45] D. A. Smirnov, Chaos **28**, 075303 (2018).
[46] D. A. Smirnov, Europhys. Lett. **128**, 20006 (2019).
[47] J. Pearl, *Causality: Models, Reasoning, and Inference* (Cambridge University Press, Cambridge, UK, 2000).
[48] A. B. Barrett and L. Barnett, Front. Neuroinf. **7**, 6 (2013).
[49] L. Barnett, A. B. Barrett, and A. K. Seth, Phys. Rev. Lett. **103**, 238701 (2009).
[50] L. Barnett and T. Bossomaier, Phys. Rev. Lett. **109**, 138105 (2012).
[51] M. Prokopenko, J. T. Lizier, and D. C. Price, Entropy **15**, 524 (2013).
[52] J. F. Restrepo, D. M. Mateos, and G. Schlotthauer, Phys. Rev. E **101**, 052117 (2020).
[53] A. Krakovska, J. Jakubik, M. Chvostekova, D. Coufal, N. Jajcay, and M. Palus, Phys. Rev. E **97**, 042207 (2018)
[54] M. Palus, A. Krakovska, J. Jakubik, and M. Chvostekova, Chaos **28**, 075307 (2018).
[55] A. Papana, C. Kyrtsou, D. Kugiumtzis, and C. Diks, Entropy **15**, 2635 (2013).
[56] D. A. Smirnov and R. G. Andrzejak, Phys. Rev. E **71**, 036207 (2005).
[57] M. Palus and A. Stefanovska, Phys. Rev. E **67**, 055201(R) (2003).
[58] L. Barnett, J. T. Lizier, M. Harre, A. K. Seth, and T. Bossomaier, Phys. Rev. Lett. **111**, 177203 (2013)
[59] P. Laiou and R. G. Andrzejak, Phys. Rev. E **95**, 012210 (2017).
[60] G. I. Barenblatt, *Scaling, Self-similarity, and Intermediate Asymptotics* (Cambridge University Press, Cambridge, UK, 2014).
[61] A. Kolmogoroff, Math. Ann. **104**, 415 (1931).
[62] I. I. Gihman and A. V. Skorohod, *The Theory of Stochastic Processes* (Springer, Berlin, 1975).
[63] N. Wiener, in *Modern Mathematics for Engineers*, edited by E. F. Beckenbach (McGraw–Hill, New York, 1956).
[64] C. W. J. Granger, Inform. Contr. **6**, 28 (1963).
[65] C. W. J. Granger, J. Econ. Dynam. Contr. **2**, 329 (1980).
[66] L. Arnold, *Random Dynamical Systems* (Springer-Verlag, Berlin, 1998).
[67] L. Ljung, *System identification: Theory for the User* (Prentice-Hall, Englewood Cliffs, NJ, 1987).
[68] M. F. Jansen, D. Dommenget, and N. Keenlyside, J. Climate **22**, 550 (2009).
[69] S. L. Bressler and A. K. Seth, NeuroImage **58**, 323 (2011).
[70] K. Sekimoto, *Stochastic Energetics*, Lect. Notes Phys. 799 (Springer, Berlin, 2010).
[71] L. Barnett and A. K. Seth, J. Neurosci. Meth. **275**, 93 (2017).
[72] K. Hasselmann, Tellus **28**, 473 (1976).
[73] T. J. Crowley, Science **289**, 270 (2000).
[74] A. M. Yaglom, *An Introduction to the Theory of Stationary Random Functions* (Prentice-Hall, Englewood Cliffs, NJ, 1962).
[75] V. S. Pugachev and I. N. Sinitsyn, *Stochastic Differential Systems: Analysis and Filtering* (Wiley, Chichester, UK, 1987).
[76] Since strong inequalities are considered as typical situations, there are small (infinitesimal) terms in almost all obtained expressions. These terms are omitted throughout the paper. In the present example, such an omitted term is of the order of $\max\{1/M, M/L\}$ relatively to the presented approximate value, since the exact expression is given by Eq. (7).
[77] If one has a full set of autoregressive model coefficients whose number is not very small, e.g., about 10, then the relationships between the coupling quantifiers may still be useful to estimate the coupling-on DCE. Indeed, if the time series is short, model coefficients estimates are not so reliable. Hence, direct estimation of the long-term DCEs through the model manipulations is possible, but its reliability is also questioned. If sample correlation functions are close to such functions of the system (6), e.g., both ACFs are exponentially decaying and the CCF exhibits a single extremum, then it is reasonable to suppose

that the system (6) is a rough model capturing the basic, most reliable properties of the processes under study. The rough and simple *S*-*T* relationships obtained for the system (6) may then appear even more justified than the empirical model-based estimates.

[78] D. A. Smirnov, N. Marwan, S. F. M. Breitenbach, F. Lechleitner, and J. Kurths, Europhys. Lett. **117**, 10004 (2017).
[79] D. A. Smirnov, S. F. M. Breitenbach, G. Feulner, F. A. Lechleitner, K. M. Prufer, J. U. L. Baldini, N. Marwan, and J. Kurths, Sci. Rep. **7**, 11131 (2017).
[80] I. I. Mokhov and D. A. Smirnov, Izv. Atmos. Oceanic Phys. **44**, 263 (2008).
[81] I. I. Mokhov and D. A. Smirnov, Dokl. Earth Sci. **427**, 798 (2009).
[82] I. I. Mokhov, D. A. Smirnov, P. I. Nakonechny, S. S. Kozlenko, Ye. P. Seleznev, and J. Kurths, Geophys. Res. Lett. **38**, L00F04 (2011).
[83] M. G. Rosenblum and A. S. Pikovsky, Phys. Rev. E **64**, 045202(R) (2001).
[84] D. A. Smirnov and B. P. Bezruchko, Phys. Rev. E **68**, 046209 (2003).
[85] B. Kralemann, L. Cimponeriu, M. Rosenblum, A. Pikovsky, and R. Mrowka, Phys. Rev. E **76**, 055201(R) (2007).
[86] B. P. Bezruchko, V. I. Ponomarenko, M. D. Prokhorov, D. A. Smirnov, and P. A. Tass, Phys. Usp. **51**, 304 (2008).
[87] D. A. Smirnov and B. P. Bezruchko, Phys. Rev. E **79**, 046204 (2009).
[88] Z. Levnajic and A. Pikovsky, Phys. Rev. Lett. **107**, 034101 (2011).
[89] B. Kralemann, M. Rosenblum, and A. Pikovsky, Chaos **21**, 025104 (2011).
[90] Y. F. Suprunenko, P. T. Clemson, and A. Stefanovska, Phys. Rev. Lett. **111**, 024101 (2013).
[91] D. Marinazzo, M. Pellicoro, and S. Stramaglia, Phys. Rev. E **73**, 066216 (2006).
[92] D. Marinazzo, M. Pellicoro, and S. Stramaglia, Phys. Rev. Lett. **100**, 144103 (2008).
[93] M. V. Sysoeva, E. Sitnikova, I. V. Sysoev, B. P. Bezruchko, and G. van Luijtelaar, J. Neurosci. Meth. **226**, 33 (2014).
[94] B. Wahl, U. Feudel, J. Hlinka, M. Wachter, J. Peinke, and J. A. Freund, Phys. Rev. E **93**, 022213 (2016).
[95] G. Sugihara, R. May, H. Ye, C. Hsieh, E. Deyle, M. Fogarty, and S. Munch, Science **338**, 496 (2012).
[96] M. C. Romano, M. Thiel, J. Kurths, and C. Grebogi, Phys. Rev. E **76**, 036211 (2007).
[97] J. H. Feldhoff, R. V. Donner, J. F. Donges, N. Marwan, and J. Kurths, Phys. Lett. A **376**, 3504 (2012).
[98] J. Arnhold, K. Lehnertz, P. Grassberger, and C. E. Elger, Physica D **134**, 419 (1999)
[99] D. Chicharro and R. G. Andrzejak, Phys. Rev. E **80**, 026217 (2009).
[100] R. G. Andrzejak and T. Kreuz, Europhys. Lett. **96**, 50012 (2011).
[101] I. Malvestio, T. Kreuz, and R. G. Andrzejak, Phys. Rev. E **96**, 022203 (2017).
[102] D. Harnack, E. Laminski, M. Schunemann, and K. R. Pawelzik, Phys. Rev. Lett. **119**, 098301 (2017).
[103] J. M. Amigo and Y. Hirata, Chaos **28**, 075302 (2018).
[104] I. V. Sysoev, M. D. Prokhorov, V. I. Ponomarenko, and B. P. Bezruchko, Phys. Rev. E **89**, 062911 (2014).
[105] I. V. Sysoev, V. I. Ponomarenko, D. D. Kulminskiy, and M. D. Prokhorov, Phys. Rev. E **94**, 052207 (2016).
[106] I. V. Sysoev, V. I. Ponomarenko, and M. D. Prokhorov, Nonlin. Dynam. **95**, 2103 (2019).
[107] D. Mukhin, A. Gavrilov, A. Feigin, E. Loskutov, and J. Kurths, Sci. Rep. **5**, 15510 (2015).
[108] A. Gavrilov, D. Mukhin, E. Loskutov, E. Volodin, A. Feigin, and J. Kurths, Chaos **26**, 123101 (2016)
[109] B. Schelter, J. Timmer, and M. Eichler, J. Neurosci. Methods **179**, 121 (2009).
[110] L. Faes, G. Nollo, S. Stramaglia, and D. Marinazzo, Phys. Rev. E **96**, 042150 (2017).
[111] L. Barnett and A. K. Seth, Phys. Rev. E **91**, 040101(R) (2015).
[112] J. Geweke, J. Am. Stat. Assoc. **77**, 304 (1982).