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Transient and equilibrium causal effects in coupled oscillators

Dmitry A. Smirnov[a)]
Saratov Branch, V.A. Kotel'nikov Institute of Radio Engineering and Electronics, Russian Academy of Sciences, 38 Zelyonaya Street, Saratov 410019, Russia

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Two quite different types of causal effects are given by (i) changes in near future states of a driven system under changes in a current state of a driving system and (ii) changes in statistical characteristics of a driven system dynamics under changes in coupling parameters, e.g., under switching the coupling off. The former can be called transient causal effects and can be estimated from a time series within the well established framework of the Wiener–Granger causality, while the latter represent equilibrium (or stationary) causal effects which are often most interesting but generally inaccessible to estimation from an observed time series recorded at fixed coupling parameters. In this work, relationships between the two kinds of causal effects are found for unidirectionally coupled stochastic linear oscillators depending on their frequencies and damping factors. Approximate closed-form expressions for these relationships are derived. Their limitations and possible extensions are discussed, and their practical applicability to extracting equilibrium causal effects from time series is argued. *Published by AIP Publishing.* https://doi.org/10.1063/1.5017821

Detection of causal[1–11] couplings within a complex system is important in various fields, and the Wiener–Granger causality[12,13] approach including its linear and nonlinear versions[14,15] is well established and widely used for coupling estimation from observed time series. This approach can be argued to concern so-called transient causal effects[10,16] and cannot assure that a detected coupling is dynamically influential. In many problems, it is most important to learn how strong and influential is the detected coupling for the "entire" observed dynamics, e.g., what would change in the driven system dynamics if the coupling was switched off,[16,17] a question similar to those asked in bifurcation analysis. Coupling characteristics of such kind are called "equilibrium causal effects" below. There are no general ways of estimating equilibrium causal effects from time series, but one may hope to perform such estimation having found relationships between the two kinds of causal effects for a restricted class of systems.[16] In this work, such relationships and approximate closed-form expressions for them are found for stochastic linear damped oscillators, giving an opportunity to assess equilibrium causal effects from time series in a variety of practical situations.

I. INTRODUCTION

Owing to the general importance of detecting and quantifying directional, or causal[1–11] couplings between time-evolving systems from time series, many approaches to coupling characterization have been[12–15] and are still being suggested within the frameworks of information theory,[9,11,18–28] nonlinear dynamics,[29–48] and linear stochastic processes.[49–54] The natural disciplines where these approaches are most often required and applied seem to be neuroscience[7,8,14,26–28,34,43–45,47,49–53,55] and climate science,[2,16,17,21,22,54–58] but other applications exist and range from chemistry[38] to ecology.[46]

In studies of coupled systems from their time series, one often observes that some coupling characteristics can be readily estimated from the data, but not of direct interest for a problem at hand, while those of primary interest are unavailable through time series estimation. To the first group, one can often ascribe transient causal effects representing the short-term response of a driven system state to a change in a driving system state. They can be characterized with the Wiener–Granger causality as argued in Refs. 10,16 including the celebrated transfer entropy[18,59–61] and taking into account possible difficulties of its interpretation.[62–64] The second group often implies an assessment of an overall physical (or dynamical) significance of the coupling, i.e., understanding of how important the coupling is for maintaining observed equilibrium characteristics of a driven system regime, e.g., mean-squared amplitude of oscillations.[16] Since a time series is usually recorded at fixed values of system parameters, it does not directly contain any information on what happens if the coupling is switched off, so the desired characteristics from the second group cannot be extracted from such time series. Accordingly, numerical values of the Wiener–Granger causality measures cannot be directly interpreted in terms of whether coupling is strong or weak in the sense of its influence on dynamics.

One may hope that coupling characteristics from the two groups can be related to each other for restricted, but reasonably wide classes of systems. Finding such relationships would open an exciting opportunity to extract seemingly unavailable information about the dynamical significance of coupling using well established estimates of transient causal effects. Moreover, it would allow meaningful interpretation of numerical values of the Wiener–Granger causality estimates.

[a)] Electronic mail: smirnovda@yandex.ru

A simple relationship between normalized short-term prediction improvement (a measure of the Wiener–Granger causality) and relative change in equilibrium mean-squared amplitude due to coupling removal was previously found[16] for unidirectionally coupled relaxation systems (overdamped oscillators) with one-dimensional state spaces. That class of systems seems too narrow, so it is highly desirable to extend those results and obtain more general relationships for broader classes with richer dynamical properties. This is the purpose of the present work which suggests an essential generalization to linear stochastic oscillators with two-dimensional state spaces.

A note concerning the terminology seems to be in order here. The term "transient effect" can be also quite reasonably used for dynamical systems whose parameters change in time inducing a transient process to a new equilibrium distribution of states. This meaning is not used here. Wiener–Granger causality and "transient causal effects" considered below are defined for systems with constant parameters, so there could be reasons to call them "equilibrium" rather than "transient." However, the term "transient" shows here that a change in an initial state of the systems under study induces stronger or weaker changes in the near-future distributions of states finally evolving to the same stationary distribution. The term "equilibrium causal effect" is used here to highlight another property that a change in coupling coefficient changes stationary (*equilibrium*) distribution of states. Thus, the terms "transient" and "equilibrium" here show *what* is influenced by coupling, while another possible terminology mentioned above reflects whether system parameters are constant or time-dependent. Despite this ambiguity, I believe that the terminology suggested here is useful in an appropriate context because it sheds new light on the relationships between quite different coupling characteristics and makes interpretations of their numerical values more accurate and meaningful. To avoid possible confusion, it seems sufficient to indicate explicitly in which sense the terms are used as it is done here.

Causal effects under study are defined in Sec. II. Model systems and design of the study are described in Sec. III. The obtained relationships and their approximate closed-form expressions are presented in Sec. IV. Practical applicability of the obtained relationships and their limitations and extensions (including nonlinear systems) are discussed in Sec. V. Section VI concludes.

II. TRANSIENT AND EQUILIBRIUM CAUSAL EFFECTS

Transient causal effects have been defined for state space systems in terms of "intervention–dynamical effect" as follows.[10] Let systems X and Y be specified by stochastic differential equations:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}_X(\mathbf{x}) + c_{XY}\mathbf{g}_X(\mathbf{x},\mathbf{y}) + \xi_X,\\ \dot{\mathbf{y}} &= \mathbf{f}_Y(\mathbf{y}) + c_{YX}\mathbf{g}_Y(\mathbf{y},\mathbf{x}) + \xi_Y,\end{aligned} \tag{1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are state vectors of the systems X and Y, respectively, ξ_X and ξ_Y are mutually independent Gaussian white noises, the functions $\mathbf{f}_X$ and $\mathbf{f}_Y$ represent individual dynamics of the systems, $\mathbf{g}_X$ and $\mathbf{g}_Y$ are coupling functions, and c_{XY} and c_{YX} are coupling coefficients ($c_{XY} = 0$ in this work). Let scalar observables x and y be single-valued functions of the respective states $\mathbf{x}$ and $\mathbf{y}$ and denote as $\rho_t(y\,|\mathbf{x}_0,\mathbf{y}_0)$ conditional probability density of y at time t, given an initial state $(\mathbf{x}_0,\mathbf{y}_0)$ at time $t_0 = 0$. A transient causal effect $X \to Y$ with respect to the variable y has been defined through a change in $\rho_t(y\,|\mathbf{x}_0,\mathbf{y}_0)$ occurring if an initial state of X is changed from $\mathbf{x}_0$ to $\mathbf{x}_0^*$, given $\mathbf{y}_0$. In a simple version, relative transient effect reads[16]

$$F^2_{X\to Y}(t) = \left\langle \frac{\{E[y(t)\,|\mathbf{x}_0,\mathbf{y}_0] - E[y(t)\,|\mathbf{x}_0^*,\mathbf{y}_0]\}^2}{\mathrm{var}[y(t)\,|\mathbf{x}_0,\mathbf{y}_0] + \mathrm{var}[y(t)\,|\mathbf{x}_0^*,\mathbf{y}_0]} \right\rangle_{\mathbf{x}_0,\mathbf{x}_0^*,\mathbf{y}_0}, \tag{2}$$

which is the squared difference between conditional means expressed in units of the respective conditional variances to quantify separation of the conditional distributions. Angle brackets denote averaging over stationary distribution of $\mathbf{y}_0$ while $\mathbf{x}_0$ and $\mathbf{x}_0^*$ are independently drawn from conditional stationary distribution, given $\mathbf{y}_0$. Such a change in the state of X has been called "intervention"[1–5,7,8] or "state space intervention."[10] Such an effect is a kind of "orbital" or "transient" effects[10] since it refers to non-established, finite-time conditional expectations and variances. Specifically, the causal effect (2) shows how far an ensemble (a beam) of phase orbits $\mathbf{y}(t)$ emanating from $\mathbf{y}_0$ and projected to the variable y shifts in response to the intervention in the state of X.

For the stochastic system (1), the quantity $F^2_{X\to Y}(t)$ is typically small for small response time t, reaches its maximal value for some intermediate t close to characteristic time scales of X and Y, and quickly decreases at greater response times.[10] Denote that maximal value $F^2_{X\to Y,\max} = \sup\limits_{t>0} F^2_{X\to Y}(t)$ and the respective response time $\tau_{X\to Y} = \arg\sup\limits_{t>0} F^2_{X\to Y}(t)$. Since such measures relate to finite-time responses, they were also called "short-term causal effects" and found to be well approximated by normalized prediction improvement (Wiener–Granger causality measure).[16] Therefore, $F^2_{X\to Y}(t)$ and $F^2_{X\to Y,\max}$ can often be estimated from a time series via the well established Wiener–Granger causality estimation techniques.

To characterize an "overall contribution" of the coupling $X \to Y$ to the observed dynamics of Y, one can determine changes in statistical properties of the process y which would occur if the coupling $X \to Y$ was switched off. In a simple version, one can consider stationary (equilibrium) variance of y. Denote $\sigma_y^2(c_{XY},c_{YX})$ this variance at given values of coupling coefficients c_{XY} and c_{YX}. Denote $\sigma_y^2(c_{XY},0) = \sigma_{y,0}^2$ this variance in case of suppressed influence $X \to Y$ ($c_{YX} = 0$). Then, the contribution of the coupling $X \to Y$ to the variance $\sigma_y^2(c_{XY},c_{YX})$ is defined as[16]

$$S_{X\to Y} = \frac{\sigma_y^2(c_{XY},c_{YX}) - \sigma_{y,0}^2}{\sigma_{y,0}^2}. \tag{3}$$

This characteristic belongs to the family "parametric intervention–stationary effect."[10] Since it quantifies a change in a stationary statistic which manifests itself only in the long-term behavior, i.e., over a time interval including many characteristic time scales of the systems, it was also called "long-term causal effect."[16] Here, the term "equilibrium causal effect" (in the sense of equilibrium probability distribution) is used as probably more accurate and convenient.

Apart from variance, any other statistical characteristics (e.g., higher-order moments or Shannon entropy of single-time or multiple-time stationary distributions) may well be used, especially for nonlinear systems where variance may appear non-informative as discussed in Sec. V B. However, for unidirectionally coupled linear systems (Sec. IV) and weakly nonlinear oscillators far from synchronization regime (Sec. V B), the stationary variance is a quantity which depends monotonously on a coupling coefficient and is simply computable. So the choice of the stationary variance to define a basic characteristic of an equilibrium causal effect seems to be justified.

There is no general way to estimate $S_{X\to Y}$ from an observed time series of x and y. One such possibility could be based on relating $S_{X\to Y}$ to more readily estimated transient causal effects. Having estimated $F^2_{X\to Y}(\Delta t)$ at small Δt and knowing the value of $f_{X\to Y} = \frac{d}{dt}\left(\frac{F^2_{X\to Y}(t)}{S_{X\to Y}}\right)\Big|_{t=0}$, which can be called "very short-term response" rate, one gets $S_{X\to Y} \approx F^2_{X\to Y}(\Delta t)/(f_{X\to Y}\Delta t)$. This relationship is valid at $\Delta t \ll f^{-1}_{X\to Y}$ and depends on Δt, being in this sense subjective. A more objective characteristic is based on the maximal effect $F^2_{X\to Y,\max}$ and, therefore, of a special interest is the ratio $r_{X\to Y} = \frac{S_{X\to Y}}{F^2_{X\to Y,\max}}$ which is free of an arbitrary temporal parameter and shows by what number the equilibrium causal effect exceeds (or is less than) the maximal transient causal effect.

In order to supplement the previous study,[16] I introduce here a useful auxiliary concept of a "form-factor" $m_{X\to Y}$. Note that to have a more complete theoretical picture, it is desirable to learn a connection between very short-term and maximal transient causal effects. It can be specified in the form $\frac{f_{X\to Y}\tau_{X\to Y}S_{X\to Y}}{F^2_{X\to Y,\max}} = m_{X\to Y}$, where $m_{X\to Y}$ is the form-factor describing deviation of the temporal dependence $F^2_{X\to Y}(t)/S_{X\to Y}$ from a straight line $f_{X\to Y}t$ over the time interval $[0, \tau_{X\to Y}]$ [Fig. 1(a)]. As shown in Sec. IV, finding certain limit values of $m_{X\to Y}$ and $\tau_{X\to Y}$ and using relationship $r_{X\to Y} = m_{X\to Y}/(f_{X\to Y}\tau_{X\to Y})$ provide approximate closed-form expressions for $r_{X\to Y}$.

III. MODEL SYSTEMS AND DESIGN OF THE STUDY

Relationships between the transient and equilibrium causal effects are found below for unidirectionally coupled stochastically perturbed linear damped oscillators:

$$\begin{aligned} &\ddot{x} + 2\gamma_X\dot{x} + \omega_{0X}^2 x = \xi_X, \\ &\ddot{y} + 2\gamma_Y\dot{y} + \omega_{0Y}^2 y = \xi_Y + c_{YX}x, \end{aligned} \tag{4}$$

where $\mathbf{x} = (x, \dot{x})$ and $\mathbf{y} = (y, \dot{y})$ are two-dimensional state vectors, γ_X and γ_Y are damping factors, ω_{0X} and ω_{0Y} are natural frequencies, c_{YX} is the coefficient of unidirectional coupling, ξ_X and ξ_Y are mutually independent one-dimensional Gaussian white noises with autocovariance functions (ACFs) $\langle\xi_X(t_1)\xi_X(t_2)\rangle = \Gamma_{\xi,X}\delta(t_1 - t_2)$, and $\langle\xi_Y(t_1)\xi_Y(t_2)\rangle = \Gamma_{\xi,Y}\delta(t_1 - t_2)$, $\Gamma_{\xi,X}$ and $\Gamma_{\xi,Y}$ are noise intensities. If γ_X and γ_Y are less than the respective natural frequencies, the oscillators are weakly damped. This is the main situation of interest here, since it differs from the overdamped oscillators (relaxation systems) considered previously[16] and corresponding to damping factors which strongly exceed the respective frequencies. In the latter case, the oscillator equations (4) reduce to

$$\begin{aligned} &\dot{x} + \alpha_X x = \zeta_X, \\ &\dot{y} + \alpha_Y y = \zeta_Y + \tilde{c}_{YX}x, \end{aligned} \tag{5}$$

where the state vectors are one-dimensional (x and y), $\alpha_X = \omega_{0X}^2/2\gamma_X$, $\alpha_Y = \omega_{0Y}^2/2\gamma_Y$, $\tilde{c}_{YX} = c_{YX}/2\gamma_Y$, and ζ_X and ζ_Y are white noises with $\langle\zeta_X(t_1)\zeta_X(t_2)\rangle = \Gamma_{\zeta,X}\delta(t_1 - t_2)$, $\langle\zeta_Y(t_1)\zeta_Y(t_2)\rangle = \Gamma_{\zeta,Y}\delta(t_1 - t_2)$, $\Gamma_{\zeta,X} = \Gamma_{\xi,X}/4\gamma_X^2$, and $\Gamma_{\zeta,Y} = \Gamma_{\xi,Y}/4\gamma_Y^2$.

For the sake of methodological clarity and systematic development of the formulas for $r_{X\to Y}$, Sec. IV A starts with the simplest system (5) repeating some results of Ref. 16 and providing additional information on the values of $m_{X\to Y}$. Then, X and Y are made in turn oscillatory. Namely, Sec. IV B considers the case of an oscillator driving a relaxation system:

$$\begin{aligned} &\ddot{x} + 2\gamma_X\dot{x} + \omega_{0X}^2 x = \xi_X, \\ &\dot{y} + \alpha_Y y = \zeta_Y + \tilde{c}_{YX}x. \end{aligned} \tag{6}$$

Section IV C presents the case of a relaxation system driving an oscillator:

$$\begin{aligned} &\dot{x} + \alpha_X x = \zeta_X, \\ &\ddot{y} + 2\gamma_Y\dot{y} + \omega_{0Y}^2 y = \xi_Y + c_{YX}x. \end{aligned} \tag{7}$$

Section IV D considers coupled oscillators (4). To compare results for any of the systems (4), (6), or (7) with a simpler analogue, an oscillator can be replaced by a respective (i.e., reasonably well approximating) relaxation system. The latter is defined here as follows: say, for X: $\alpha_X = \omega_{0X}^2/2\gamma_X$ for $\gamma_X/\omega_{0X} > 1$, $\alpha_X = \gamma_X$ for $\gamma_X/\omega_{0X} < 1/2$, and $\alpha_X = \omega_{0X}/2$ for $1/2 \le \gamma_X/\omega_{0X} \le 1$.

All the quantities of interest, i.e., $r_{X\to Y}$, $\tau_{X\to Y}$, $m_{X\to Y}$, and $f_{X\to Y}$, are determined versus the parameters of the oscillators which are varied in a wide range covering at least two orders of magnitude. Results depend only on non-dimensional ratios of parameters and are expressed via α_X/α_Y in Sec. IV A, γ_X/ω_{0X} and α_Y/ω_{0X} in Sec. IV B, γ_Y/ω_{0Y} and α_X/ω_{0Y} in Sec. IV C, and γ_X/ω_{0X}, γ_Y/ω_{0X}, and ω_{0Y}/ω_{0X} in Sec. IV D. To determine the quantities of interest, linear ordinary differential equations for the first and second conditional moments and linear algebraic equations for the stationary moments are solved[10] as summarized in Appendix, $f_{X\to Y}$ and $S_{X\to Y}$ being always found explicitly. In addition to the methods of previous studies,[10,16] asymptotic behavior of all these quantities with diminishing damping factors is studied and approximating formulas for $r_{X\to Y}$ are derived.

The quantity $S_{X\to Y}$ is quadratic with respect to the coupling coefficient c_{YX} for the class of systems considered. $F^2_{X\to Y,\max}$ is also quadratic if c_{YX} is small enough. Therefore, the ratio $r_{X\to Y}$ is independent of c_{YX}, if c_{YX} (or the respective $S_{X\to Y}$) is only moderately large. In particular, this independence holds true to a typical error of the order of 1% if $S_{X\to Y} < 0.1$ (Sec. V A). For each set of parameter values

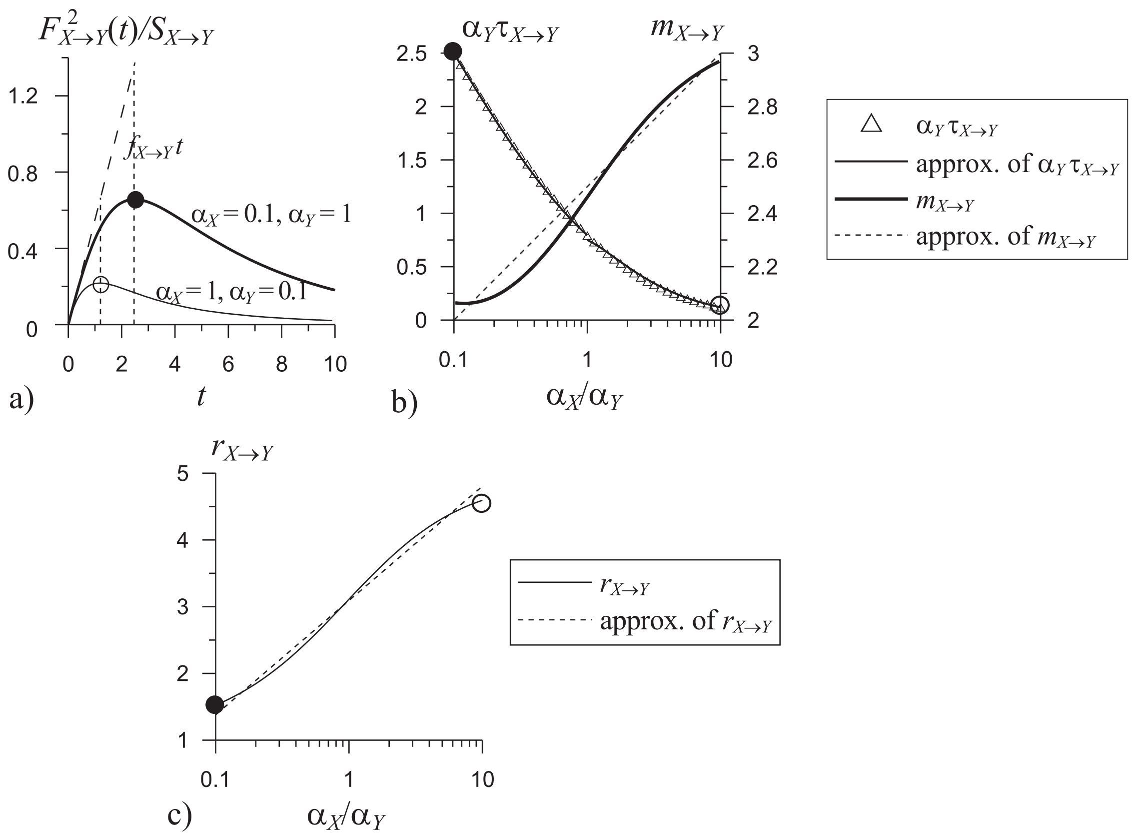

FIG. 1. Causal effects in coupled relaxation systems (5): (a) Relative transient causal effects versus response time for two sets of parameter values, open and filled circles indicate maxima of the plots, long dashes show the extrapolated very short-term effect $f_{X\to Y}t$, vertical dashed lines guide an eye to see the distances from both plots to the extrapolated straight line; (b) the normalized maximum response time $\alpha_Y\tau_{X\to Y}$ [triangles, left axis, the open and filled circles correspond to those in Fig. 1(a)] with two pieces of its analytic approximation (thin solid lines, Sec. IV A) and the form-factor $m_{X\to Y}$ (thick solid line, right axis) with its analytic approximation (dashed line, Sec. IV A) versus the ratio of relaxation rates; (c) the equilibrium-to-transient effect ratio $r_{X\to Y}$ [solid line, the open and filled circles correspond to those in Fig. 1(a)] with its analytic approximation (8) (dashed line) versus the ratio of relaxation rates.

below, c_{YX} is specified so to provide $S_{X\to Y} = 0.01$. Extensive checks confirmed that the results for $S_{X\to Y} = 0.1$ are almost indistinguishable, and even $S_{X\to Y}$ up to 0.5 most often leads to quite moderate changes in $r_{X\to Y}$.

A brief summary of the numerical results is presented in Table I for convenience. It can be seen in advance that the magnitude of $r_{X\to Y}$ (which is the main quantity of interest) covers broad ranges of possible values from those much less than unity (0.03) to those considerably greater than unity (4.9) for the reasonable intervals of parameter values studied. To give an idea about the meaning of the concrete numerical values of $r_{X\to Y}$, I note that $r_{X\to Y} \approx 5$ shows that seemingly small maximal prediction improvement (Wiener–Granger causality measure) about several percents (e.g., $F^2_{X\to Y,\max} = 0.05$) corresponds to quite noticeable equilibrium causal effect ($S_{X\to Y} = 0.25$, i.e., stationary variance increases by a quarter due to the presence of coupling). Hence, for some systems so small prediction improvements cannot be ignored as insignificant. An opposite situation is also possible, e.g., $r_{X\to Y} \approx 0.03$ shows that a quite large prediction improvement of 30% corresponds to a very small equilibrium causal effect of $S_{X\to Y} = 0.01$, i.e., 1%. Thus, for some systems even large prediction improvements correspond to couplings whose overall contribution to the variance is negligibly small. It is thus important to learn conditions for both situations in order to be able to interpret the Wiener–Granger causality estimates properly.

IV. ANALYTIC AND NUMERICAL RESULTS

A. Coupled relaxation systems

The equilibrium causal effect for the system (5) reads $S_{X\to Y} = c^2_{YX}\sigma^2_{x,0}/[2\alpha_Y(\alpha_X+\alpha_Y)\sigma^2_{y,0}]$ and the response rate is $f_{X\to Y} = (\alpha_X+\alpha_Y)/2$. Relative transient causal effects are presented in Fig. 1(a) for two cases: a fast system driving a slow one ($\alpha_X = 1, \alpha_Y = 0.1$) and vice versa ($\alpha_X = 0.1, \alpha_Y = 1$). In both cases, $f_{X\to Y}$ is the same, but the maximum response time differs: $\tau_{X\to Y} \approx 1.25/\alpha_X$ if the driving system is faster and $\tau_{X\to Y} \approx 2.3/\alpha_Y$ if the driven system is faster [Figs. 1(a) and 1(b)]. The form-factor also differs: in the first case, the extrapolated value of the very short-term effect $f_{X\to Y}\tau_{X\to Y}$ exceeds the value of $F^2_{X\to Y}(\tau_{X\to Y})/S_{X\to Y}$ by $m_{X\to Y} \approx 3$ times and in the second case by $m_{X\to Y} \approx 2$ times [Figs. 1(a) and 1(b)]. Finally, one gets $r_{X\to Y} \approx 4.5$ in the first case and $r_{X\to Y} \approx 1.5$ in the second case [Fig. 1(c)].

Overall, within a reasonable range $0.1 \le \alpha_X/\alpha_Y \le 10$ comprising both close and considerably different relaxation times, the value of $m_{X\to Y}$ rises from 2 to 3 almost linearly in $\ln(\alpha_x/\alpha_y)$, the relative error of such approximation being less than 5% [Fig. 1(b), dashed and thick solid lines], and

$$r_{X\to Y} = 3.1 + 0.75\ln(\alpha_X/\alpha_Y), \tag{8}$$

with the relative error less than 10% [Fig. 1(c)]. The maximum response time [Fig. 1(b), triangles] is accurately

TABLE I. Brief summary of the numerical values obtained for the causal effects under study. The first column—model systems. The second column—ranges of parameter values checked. The third column—ranges of the obtained values of the equilibrium-to-transient causal effects ratio $r_{X\to Y}$. The fourth column—maximum response time $\tau_{X\to Y}$. The fifth column—form-factor $m_{X\to Y}$.

Systems under study	Parameters values presented	$r_{X\to Y}$	$\tau_{X\to Y}$	$m_{X\to Y}$
Relaxation system X drives relaxation system Y (5)	$0.1 \le \alpha_X/\alpha_Y \le 10$	$1.5-4.6$	$(0.12-2.5)/\alpha_Y$	$2.1-3.0$
Oscillator X drives relaxation system Y (6)	$0.1 \le \alpha_Y/\omega_{0X} \le 10, 0.01 \le \gamma_X/\omega_{0X} \le 100$	$0.17-3.5$	$(0.43-30.0)/\omega_{0X}$	$1.6-4.0$
Relaxation system X drives oscillator Y (7)	$0.1 \le \alpha_X/\omega_{0Y} \le 10, 0.01 \le \gamma_Y/\omega_{0Y} \le 100$	$0.13-4.9$	$(0.13-12.0)/\omega_{0Y}$	$1.2-3.3$
Oscillator X drives oscillator Y (4)	$0.01 \le \gamma_X/\omega_{0X} \le 100, 0.1 \le \gamma_Y/\omega_{0Y} \le 10, 0.2 \le \omega_{0Y}/\omega_{0X} \le 5$	$0.03-4.5$	$(0.63-120)/\omega_{0X}$	$0.94-3.3$

approximated by two pieces [Fig. 1(b), thin solid lines]: $\tau_{X\to Y} = [0.1 + \ln(1+\alpha_Y/\alpha_X)]/\alpha_Y$ for $0.1 \le \alpha_X/\alpha_Y < 1$ (the error is less than 0.5%) and $\tau_{X\to Y} = (1.25 - 0.5\alpha_Y/\alpha_X)/\alpha_X$ for $1 < \alpha_X/\alpha_Y \le 10$ (the error is less than 5%). These rough approximations are convenient for fast practical assessments. More precise approximate expressions valid over a broader interval of α_X/α_Y are readily obtained, but appear more cumbersome, so they are not reported here.

B. An oscillator drives a relaxation system

The system (6) exhibits features both different from and common with the simpler system (5). Here and below, explicit formulas for $S_{X\to Y}$ are somewhat cumbersome and not reported. The very short-term response rate is $f_{X\to Y} = [\omega_{0X}^2/(2\gamma_X + \alpha_Y) + \alpha_Y]/2$. It is close to that of the system (5) with $\alpha_X = \omega_{0X}^2/(2\gamma_X)$ if $\gamma_X/\omega_{0X} \gg 1$ (for any α_Y). Numerical results show that $\gamma_X/\omega_{0X} \ge 3$ suffices for the difference in $r_{X\to Y}$ between the system (6) and the respective system (5) to be less than 6% for any α_Y/ω_{0X}. For $\gamma_X/\omega_{0X} \ge 2$, this difference is less than 12%.

Another, less expected, case of closeness in $r_{X\to Y}$ for the systems (6) and (5) is very large relaxation rate of the driven system $\alpha_Y/\omega_{0X} \gg 1$ and almost any γ_X/ω_{0X} [Fig. 2(a)]. Namely, for $\alpha_Y/\omega_{0X} \ge 10$, the difference in $r_{X\to Y}$ between the two systems is less than 10% if $\gamma_X/\omega_{0X} \ge 2$ or $\gamma_X/\omega_{0X} \le 1/6$ and less than 20% if $\gamma_X/\omega_{0X} \ge 3/2$ or $\gamma_X/\omega_{0X} \le 1/3$ [Fig. 2(d)]. Recall that in case of $\gamma_X/\omega_{0X} \le 1/3$ the respective relaxation system is that with $\alpha_X = \gamma_X$: then the maximum response times for the systems (5) and (6) are close to each other, temporal dependencies $F_{X\to Y}^2(t)/S_{X\to Y}$ over long intervals are mutually close too [Fig. 2(a)] with a superposed oscillatory component for the system (6). The closeness can be anticipated noticing $f_{X\to Y} \approx \alpha_Y/2$ for large α_Y/ω_{0X} in the system (6) similarly to the respective system (5) with $\alpha_Y \gg \alpha_X$. Thus, the property of the relaxation system with $\alpha_X = \gamma_X$ to reflect the exponentially decaying envelope of the oscillator's ACF suffices to reproduce the overall behavior of the transient causal effect in Fig. 2(d). Some difference between the systems (5) and (6) occurs for an intermediate damping $1/3 < \gamma_X/\omega_{0X} < 2$ where the relative difference in $r_{X\to Y}$ reaches 26% (and 50% between the maximum response times) for $\gamma_X/\omega_{0X} = 1/2$. This is because ACF for such an oscillator is poorly approximated by a relaxation system. Still, the difference is moderate, not by an order of magnitude.

The strongest difference between the systems (6) and (5) holds in case of a weakly damped oscillator $\gamma_X/\omega_{0X} \le 1/3$ and a slow relaxation system $\alpha_Y/\omega_{0X} \le 1/2$. The stronger the inequalities, the greater this difference [Figs. 2(c) and 2(f)]. For $\gamma_X/\omega_{0X} \ll 1$ (i.e., for narrow-band oscillator), the maximum response time tends to $\tau_{X\to Y} \approx 2/\omega_{0X}$ [Fig. 2(h)] and the form-factor to $m_{X\to Y} \approx 1.6$ [Fig. 2(i)]. The latter number can be understood as being close to the form-factor for a sine function which equals $\pi/2$. Based on these approximate limits, one can derive an explicit closed-form expression:

$$r_{X\to Y} = 1.6\left(\frac{\omega_{0X}}{2\gamma_X + \alpha_Y} + \frac{\alpha_Y}{\omega_{0X}}\right)^{-1}, \tag{9}$$

which is quite accurate in the range $\gamma_X/\omega_{0X} \le 1/3$ and $\alpha_Y/\omega_{0X} \le 1/2$ [Fig. 2(f), long dashes]. In particular, for $\gamma_X/\omega_{0X} = \alpha_Y/\omega_{0X} = 0.1$ Eq. (9) gives $r_{X\to Y} = 0.48$ which is very close to the exact $r_{X\to Y} = 0.46$. Thus, in order to determine the equilibrium causal effect one must divide the maximal transient causal effect by 2 instead of multiplying it by some number from 1.5 to 4.5 as it holds for the system (5). In the limit of infinitely narrow-band oscillations ($\gamma_X/\omega_{0X} \to 0$), for $\alpha_Y/\omega_{0X} = 0.1$ one gets $r_{X\to Y} = 0.16$, i.e., the equilibrium causal effect is even six times as small as the maximal transient causal effect. Observing that here a fast oscillatory system drives a slower relaxation system, one could erroneously guess $r_{X\to Y} \approx 5$ using a naive analogy with the system (5). This demonstrates that the case of an oscillatory driving system is quite different.

For an intermediate relaxation rate $\alpha_Y/\omega_{0X} = 1$, the formula (9) is not accurate but can be adjusted if the maximum response time is corrected to $\tau_{X\to Y} = 1.5/\omega_{0X}$ [Fig. 2(h)]. Then, the factor 1.6 in (9) should be replaced by 2.1 and the approximation still works well [Fig. 2(e), long dashes]. For other cases of intermediate relaxation rates and moderate damping [thick solid lines, circles, and triangles in Figs. 2(g)–2(i)], the ratio $r_{X\to Y}$ and other quantities of interest also cannot be approximated well by the formula (9) and are not close to those for the system (5). They take on intermediate values which can be estimated in practice via interpolation between nearby points accessible to approximations via one of the above two ways. In the specific case of $\alpha_Y/\omega_{0X} = 1$ and $\gamma_X/\omega_{0X} = 1$, one gets $r_{X\to Y} = 1.75$ quite different from $r_{X\to Y} = 2.5$ for the respective system (5).

To summarize, the equilibrium-to-transient causal effects ratio $r_{X\to Y}$ may strongly differ between the cases of relaxation (5) and oscillatory (6) driving systems, its value for the respective system (5) being always an upper bound to that for the system (6). The strongest difference occurs for a weakly damped oscillator and a slow driven system, where the approximate formula (9) applies.

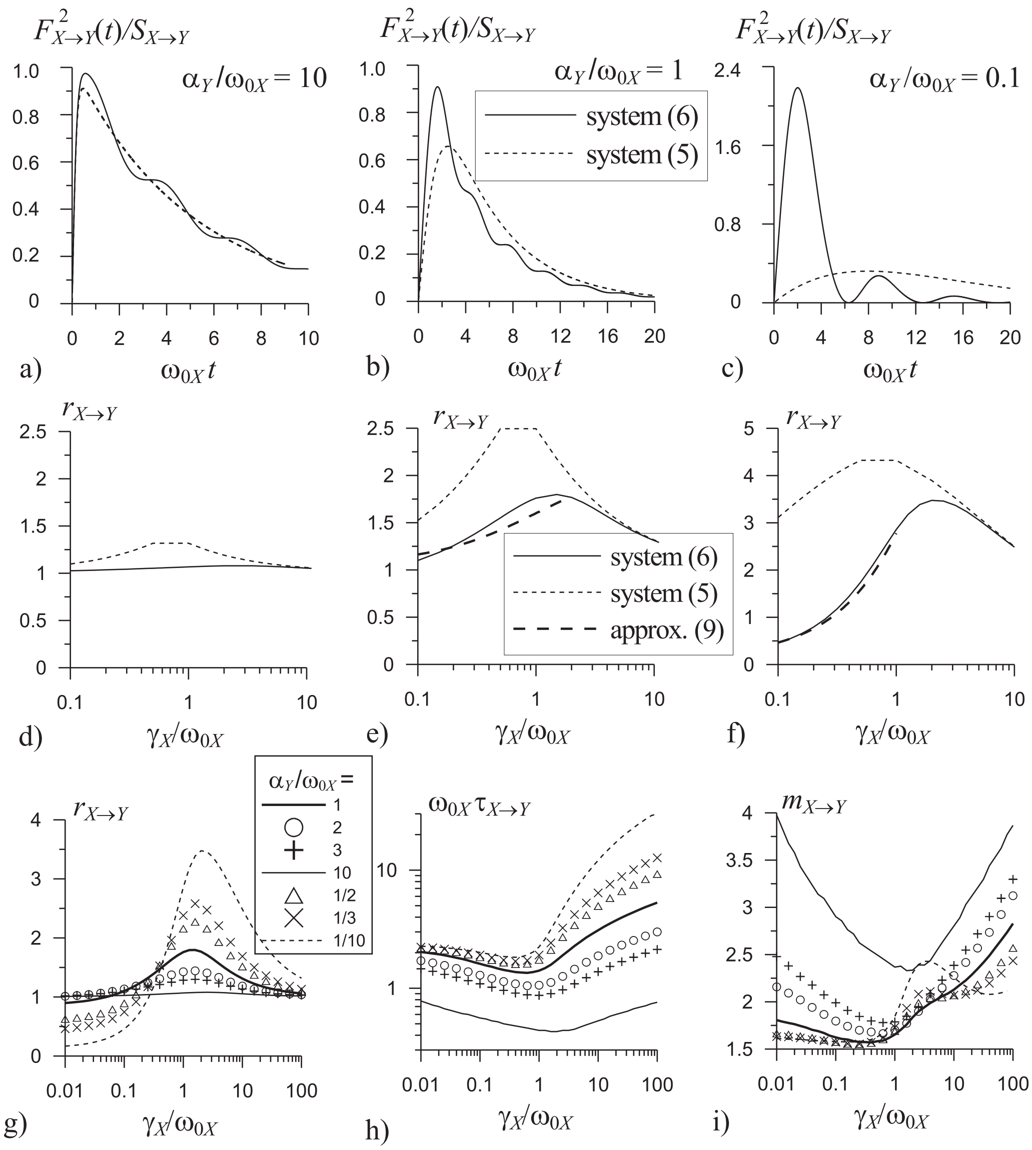

FIG. 2. Causal effects in the system (6): an oscillator drives a relaxation system. The first row shows relative transient causal effects for the system (6) with $\gamma_X/\omega_{0X} = 0.1$ (solid lines) and for the respective system (5) (dashed lines). The second row shows the ratio $r_{X\to Y}$ for the system (6) (solid lines) and for the respective system (5) (short dashes) along with the approximation (9) [long dashes, the factor 2.1 is used instead of 1.6 in Fig. 2(e)], $\alpha_Y/\omega_{0X} = 10$ in Figs. 2(a) and 2(d), $\alpha_Y/\omega_{0X} = 1$ in Figs. 2(b) and 2(e), and $\alpha_Y/\omega_{0X} = 0.1$ in Figs. 2(c) and 2(f). The third row shows $r_{X\to Y}$ (g), the relative maximum response time $\omega_{0X}\tau_{X\to Y}$ (h), and the form-factor $m_{X\to Y}$ (i) for different relaxation rates of the driven system α_Y/ω_{0X} versus damping γ_X/ω_{0X} in the driving oscillator.

C. A relaxation system drives an oscillator

The results for the system (7) appear similar to those for the system (6) with some additional complexities. The very short-term response rate is $f_{X\to Y} = (3/8)[\alpha_X + \omega_{0Y}^2/(2\gamma_Y + \alpha_X)]$, i.e., characteristics of the driving and driven systems are interchanged and an additional factor of 3/4 appears in comparison with the system (6). In the overdamped case of $\gamma_Y/\omega_{0Y} \gg 1$, one could expect a coincidence between the results for the system (7) and the respective system (5), i.e., that with $\alpha_Y = \omega_{0Y}^2/(2\gamma_Y)$. Indeed, such coincidence takes place, but under an additional condition of $\gamma_Y \gg \alpha_X$ (specifically, $\gamma_Y > 5\alpha_X$ suffices). This is because the driven system in (7) possesses two characteristic times corresponding to small relaxation rate $\omega_{0Y}^2/2\gamma_Y$ of the variable y and large relaxation rate γ_Y of the variable $\dot{y}$. The latter is damped in the free system Y, but under the influence of a very fast system X (with $\gamma_Y \le \alpha_X$) this "half-degree of freedom" of the system Y gets excited too. This makes a difference with the system (6) where that half-degree of freedom enters only the driving system and is not excited. Now, for $\gamma_Y \gg \alpha_X$ the systems (7) and (5) should be almost the same, why do the values of $f_{X\to Y}$ differ then by the factor of 3/4? This apparent contradiction is due to the fact that $f_{X\to Y}$ is defined via a derivative and so implies infinitesimally small response times t, less than any characteristic time scales of the system. The systems (7) and (5) do differ at time scales less than $1/\gamma_Y$. If the derivative in the definition of $f_{X\to Y}$ is replaced by a finite difference over a larger response time t, e.g., $5/\gamma_Y$ or greater, then numerical results show that the transient causal effects for the systems (7) and (5) coincide as expected.

Another complication is that for a very fast driving system X [$\alpha_X/\omega_{0Y} \gg 1$, Figs. 3(a) and 3(d)] the results are

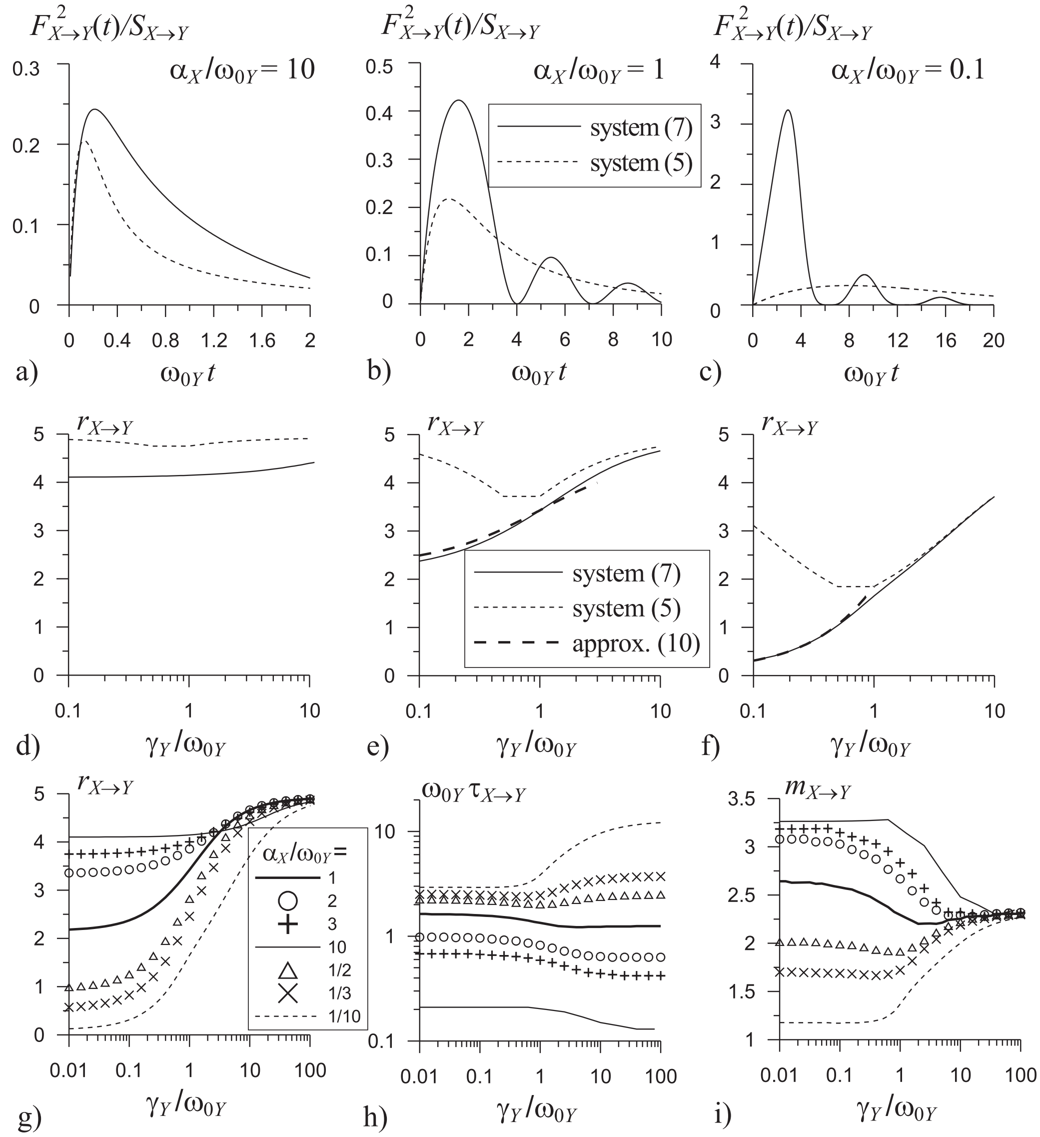

FIG. 3. Causal effects in the system (7): a relaxation system drives an oscillator. The first row shows relative transient causal effects for the system (7) with $\gamma_Y/\omega_{0Y} = 0.1$ (solid lines) and for the respective system (5) (dashed lines). The second row shows the ratio $r_{X\to Y}$ for the system (7) (solid lines) and for the respective system (5) (short dashes) along with the approximation (10) [long dashes, the factor 4.5 is used instead of 0.9 in Fig. 3(e)], $\alpha_X/\omega_{0Y} = 10$ in Figs. 3(a) and 3(d), $\alpha_X/\omega_{0Y} = 1$ in Figs. 3(b) and 3(e), and $\alpha_X/\omega_{0Y} = 0.1$ in Figs. 3(c) and 3(f). The third row shows $r_{X\to Y}$ (g), the relative maximum response time $\omega_{0Y}\tau_{X\to Y}$ (h), and the form-factor $m_{X\to Y}$ (i) for different relaxation rates of the driving system α_X/ω_{0Y} versus damping γ_Y/ω_{0Y} in the driven oscillator.

not so close to the respective relaxation systems (5) as they are for the system (6) with $\alpha_Y/\omega_{0X} \gg 1$. The difference in $r_{X\to Y}$ between the system (7) and the respective system (5) is about 20% for a weakly damped oscillator Y and about 10% for a strongly damped oscillator with $\gamma_Y/\omega_{0Y} = 10$ [Fig. 3(d)]. This should also be attributed to the excited second variable of the subsystem Y in (7) at variance with the system (6).

In other respects, the results for the system (7) are similar to those for the system (6), including small ratios $r_{X\to Y}$ for a narrow-band oscillator Y ($\gamma_Y/\omega_{0Y} \ll 1$) and a slow driving system ($\alpha_X/\omega_{0Y} \ll 1$) in Figs. 3(c) and 3(f). Limit values of $\tau_{X\to Y}$ and $m_{X\to Y}$ somewhat differ from those for the system (6): the maximum response time tends to $\tau_{X\to Y} = 3/\omega_{0Y}$ [Fig. 3(h)] and the form-factor to $m_{X\to Y} = 1$. The latter tendency is already seen in Fig. 3(i); however, one still has $m_{X\to Y} = 1.2$ even for $\alpha_X/\omega_{0Y} = 0.1$. Then, an approximate asymptotic expression reads

$$r_{X\to Y} = 0.9\left(\frac{\omega_{0Y}}{2\gamma_Y + \alpha_X} + \frac{\alpha_X}{\omega_{0Y}}\right)^{-1}. \tag{10}$$

For an infinitely narrow-band oscillator, one gets $r_{X\to Y} = 0.9\alpha_X/\omega_{0Y}$ which is almost two times as small as that for the system (6) with $\alpha_Y/\omega_{0X} = \alpha_X/\omega_{0Y}$. For $\alpha_X/\omega_{0Y} = 0.1$, an approximate value of $r_{X\to Y} = 0.09$ is reasonably close to the exact value of $r_{X\to Y} = 0.13$ [Fig. 3(g)], i.e., the maximal transient effect is 7.5 times as great as the equilibrium causal effect.

As in the previous example, the results for intermediate parameter values can be obtained via special adjustment of coefficients or via interpolation. For example, for $\alpha_X/\omega_{0Y} = 1$ [Figs. 3(b) and 3(e)] it holds $\tau_{X\to Y} = 1.4/\omega_{0X}$ [Fig. 3(h), thick line] and $m_{X\to Y} = 2.4$ [Fig. 3(i), thick line] that gives another multiplier in Eq. (10) and a good approximation in Fig. 3(e) (long dashes).

To summarize, the equilibrium-to-transient causal effects ratio $r_{X\to Y}$ may strongly differ between the cases of relaxation (5) and oscillatory (7) systems Y, its value for the respective system (5) always being an upper bound for that in the system (7). The strongest difference occurs for a weakly damped oscillator and a slow driving system, where the formula (10) is accurate.

D. Coupled oscillators

The central point of this study is the oscillators (4). The very short-term response rate reads $f_{X\to Y} = \frac{3}{16}\frac{(\omega_{0Y}^2-\omega_{0X}^2)^2+4(\gamma_X+\gamma_Y)(\gamma_X\omega_{0Y}^2+\gamma_Y\omega_{0X}^2)}{\gamma_X\omega_{0X}^2+\gamma_Y\omega_{0Y}^2+4\gamma_X\gamma_Y(\gamma_X+\gamma_Y)}$. If either $\gamma_X/\omega_{0X} \geq 3$ or $\gamma_Y/\omega_{0Y} \geq 3$, then this case reduces to the respective system (6) or (7). Therefore, the most interesting case is that of simultaneously small values of $\gamma_X/\omega_{0X} \leq 1/3$ and $\gamma_Y/\omega_{0Y} \leq 1/3$. It divides into three sub-cases according to the frequencies ratio ω_{0Y}/ω_{0X} which can be (i) much less than unity (a fast oscillator drives a slow one, left columns in Figs. 4 and 5), (ii) much greater than unity (a slow oscillator drives a fast one, right columns in Figs. 4 and 5), and (iii) about unity (close oscillation frequencies, central column in Figs. 4 and 5).

The case (i) is illustrated in Figs. 4(a) and 4(d) where $\omega_{0Y}/\omega_{0X} = 1/5$ and $\gamma_Y/\omega_{0Y} = 0.1$, and the results are compared to the respective system (6) with $\alpha_Y = \gamma_Y$. The results for the systems (4) and (6) are not strongly different, especially for $0.05 \leq \gamma_X/\omega_{0X} \leq 0.5$. Overall, the ratio $r_{X\to Y}$ is somewhat less for the system (4). Thus, for $\gamma_X/\omega_{0X} = 0.01$ it holds $r_{X\to Y} = 0.035$ [Figs. 4(d) and 5(a)] compared to $r_{X\to Y} = 0.056$ for the system (6), $\tau_{X\to Y} = 2.8/\omega_{0X}$ [Fig. 5(d)] compared to $\tau_{X\to Y} = 2.3/\omega_{0X}$, and $m_{X\to Y} = 1.6$ [Fig. 5(g)] for both systems. Using the latter two values and taking $\gamma_X/\omega_{0X} \to 0$ and $\gamma_Y/\omega_{0Y} \to 0$, one gets approximately

$$r_{X\to Y} = \frac{3\omega_{0X}(\gamma_X\omega_{0X}^2 + \gamma_Y\omega_{0Y}^2)}{(\omega_{0Y}^2 - \omega_{0X}^2)^2}. \tag{11}$$

For strongly different frequencies, it becomes $r_{X\to Y} \approx 3\left[\frac{\gamma_X}{\omega_{0X}} + \frac{\gamma_Y}{\omega_{0Y}}\left(\frac{\omega_{0Y}}{\omega_{0X}}\right)^3\right]$. If γ_X/ω_{0X} and γ_Y/ω_{0Y} are of the same order, it further simplifies to $r_{X\to Y} \approx 3\gamma_X/\omega_{0X}$. For $\gamma_X/\omega_{0X} = 0.01$ and $\gamma_Y/\omega_{0Y} = 0.1$, even the latter approximation gives a reasonable result of $r_{X\to Y} = 0.03$, while the full expression for $f_{X\to Y}$ with the corresponding asymptotic $\tau_{X\to Y}$ and $m_{X\to Y}$ [Fig. 4(d), long dashes] gives $r_{X\to Y} = 0.035$ with the error less than 1%.

The case (ii) is illustrated in Figs. 4(c) and 4(f), where $\omega_{0Y}/\omega_{0X} = 5$ and $\gamma_Y/\omega_{0Y} = 0.1$. The ratio $r_{X\to Y} = 0.18$

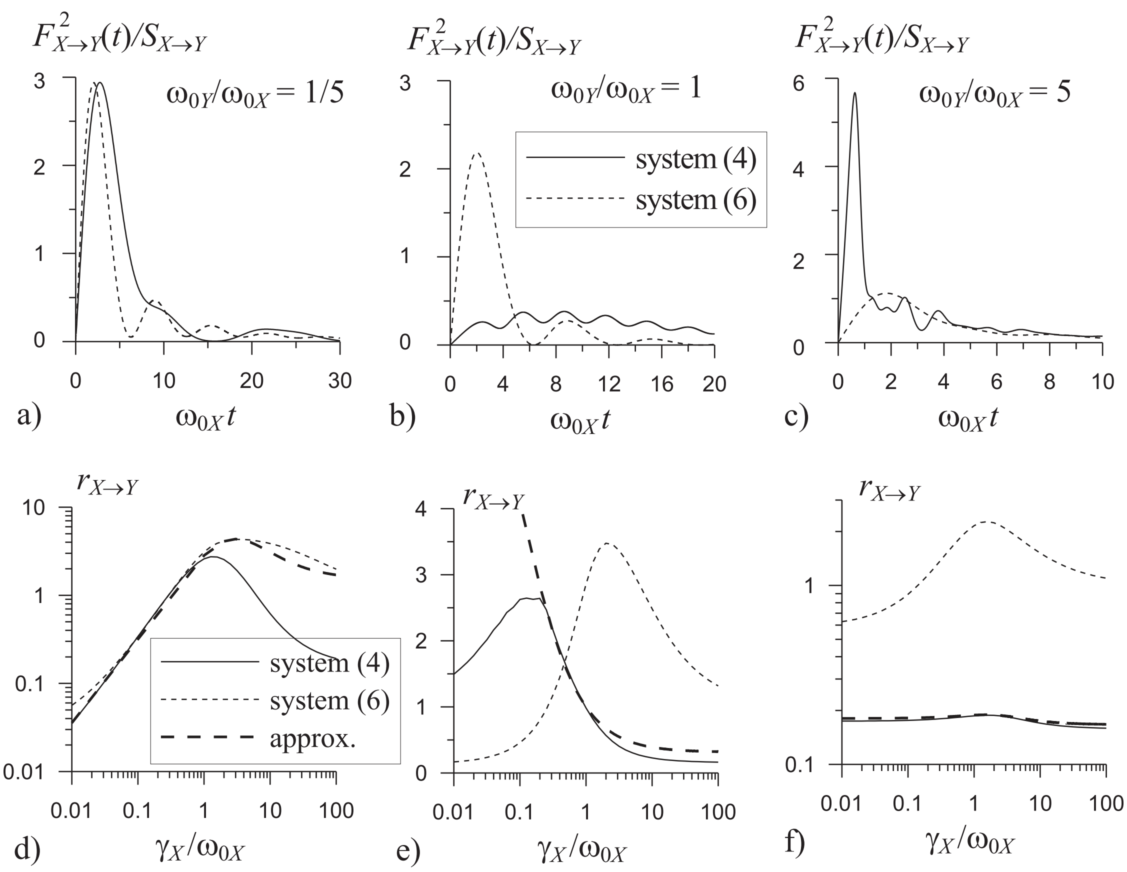

FIG. 4. Causal effects in the general system (4) with $\gamma_Y/\omega_{0Y} = 0.1$. The first row shows relative transient causal effects in the system (4) (solid lines) with $\gamma_X/\omega_{0X} = 0.1$ and for the respective system (6) (dashed lines). The second row shows the ratio $r_{X\to Y}$ for the system (4) (solid lines) and for the respective system (6) (short dashes), $\omega_{0Y}/\omega_{0X} = 1/5$ in Figs. 4(a) and 4(d), $\omega_{0Y}/\omega_{0X} = 1$ in Figs. 4(b) and 4(e), and $\omega_{0Y}/\omega_{0X} = 5$ in Figs. 4(c) and 4(f). Long dashes are approximations: Eq. (11) in Fig. 4(d), Eq. (13) in Fig. 4(e), and Eq. (12) in Fig. 4(f).

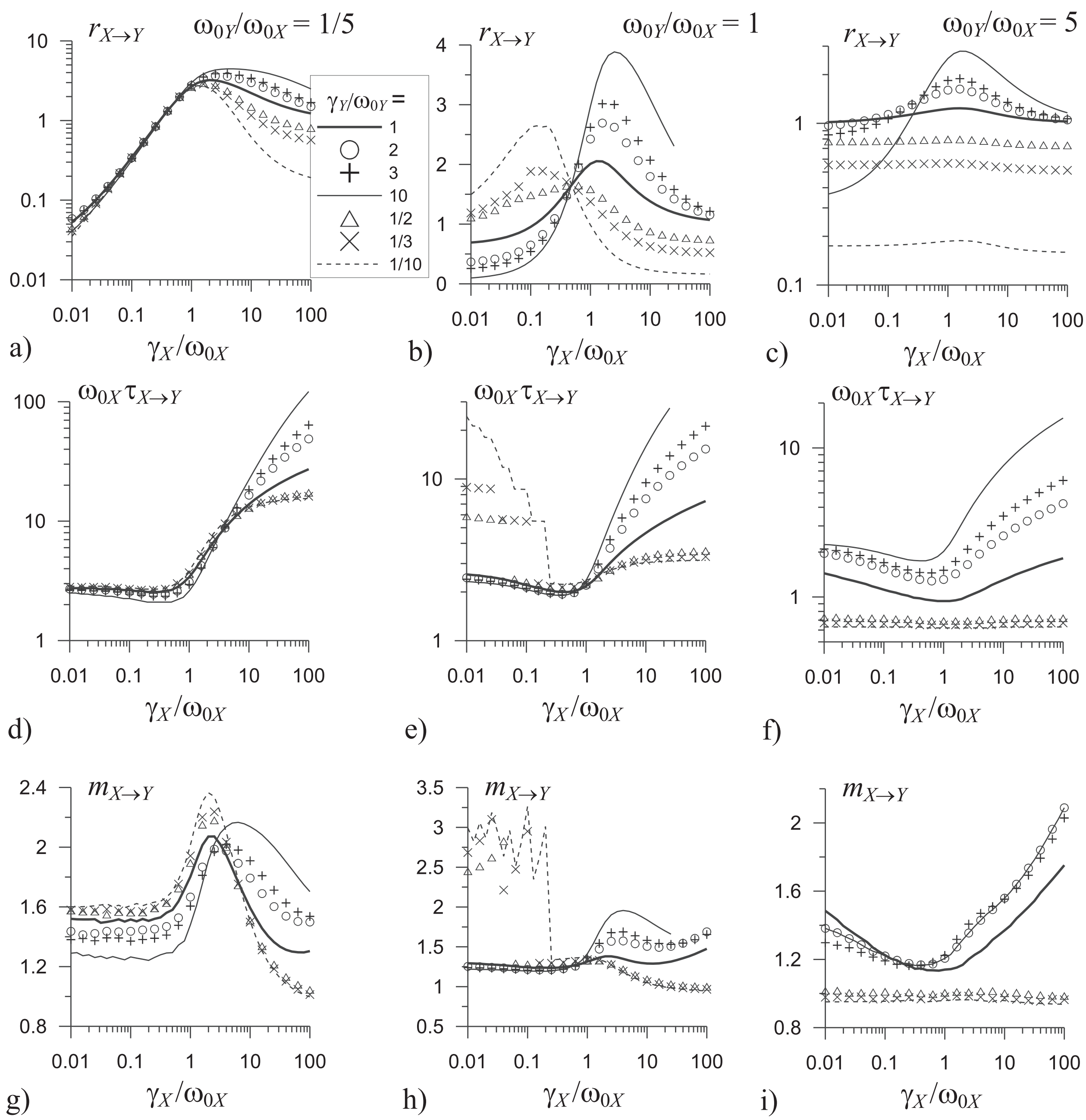

FIG. 5. Causal effects in the system of coupled oscillators (4). The first row shows the ratio $r_{X\to Y}$, the second row—the relative maximum response time $\omega_{0X}\tau_{X\to Y}$, and the third row—the form-factor $m_{X\to Y}$ for different damping factors γ_Y/ω_{0Y} of the driven oscillator [the legend in Fig. 5(a)]. The first column corresponds to the ratio of frequencies $\omega_{0Y}/\omega_{0X} = 1/5$, the second column—to $\omega_{0Y}/\omega_{0X} = 1$, and the third column—to $\omega_{0Y}/\omega_{0X} = 5$.

appears much less than that for the respective system (6), i.e., for an approximation of the envelope of the ACF of the driven oscillator using a relaxation system. In fact, $r_{X\to Y}$ varies from 0.16 to 0.19 for the entire range of γ_X/ω_{0X} in Fig. 4(f). For $\gamma_X/\omega_{0X} = 0.01$, it holds $r_{X\to Y} = 0.17$ [Figs. 4(f) and 5(c)], $\tau_{X\to Y} = 0.63/\omega_{0X}$ [half the period of the driven oscillator, Fig. 5(f)], and $m_{X\to Y} = 0.95$ [Fig. 5(i)]. Using the latter two values, $\gamma_X/\omega_{0X} \to 0$ and $\gamma_Y/\omega_{0Y} \to 0$, one gets approximately [Fig. 4(f), long dashes]

$$r_{X\to Y} = \frac{8\omega_{0X}(\gamma_X\omega_{0X}^2 + \gamma_Y\omega_{0Y}^2)}{(\omega_{0Y}^2 - \omega_{0X}^2)^2}. \quad (12)$$

For strongly different frequencies, it becomes $r_{X\to Y} \approx \frac{8\omega_{0X}}{\omega_{0Y}}\left[\frac{\gamma_Y}{\omega_{0Y}} + \frac{\gamma_X}{\omega_{0X}}\left(\frac{\omega_{0X}}{\omega_{0Y}}\right)^3\right]$. For close relative bandwidths, it further reduces to $r_{X\to Y} \approx 8\gamma_Y\omega_{0X}/\omega_{0Y}^2$. For $\gamma_X/\omega_{0X} = 0.01$ and $\gamma_Y/\omega_{0Y} = 0.1$, even the latter formula gives an accurate approximate value of $r_{X\to Y} = 0.16$.

The case (iii) is illustrated in Figs. 4(b) and 4(e) where $\omega_{0Y} = \omega_{0X}$ and $\gamma_Y/\omega_{0Y} = 0.1$. It is very different from both cases (i) and (ii). The difference is two-fold. First, the ratio $r_{X\to Y}$ increases with decreasing $\gamma_X/\omega_{0X} < 1$ while γ_X/ω_{0X} is moderately small (greater than 0.2). This can be explained using Figs. 5(e) and 5(h) where for intermediate $0.25 \le \gamma_X/\omega_{0X} \le 0.6$ and under decreasing γ_Y/ω_{0Y} down to 0.1 the maximum response time stabilizes at $\tau_{X\to Y} \approx 2.2/\omega_{0X}$ [Fig. 5(e)] and the form-factor at $m_{X\to Y} \approx 1.3$ [Fig. 5(h)]. Hence, one gets an approximation [Fig. 4(e), long dashes]

$$r_{X\to Y} = \frac{0.8(\gamma_X\omega_{0X}^2 + \gamma_Y\omega_{0Y}^2)}{(\omega_{0Y}^2 - \omega_{0X}^2)^2 + 4(\gamma_X + \gamma_Y)(\gamma_X\omega_{0Y}^2 + \gamma_Y\omega_{0X}^2)}, \tag{13}$$

reducing to $r_{X\to Y} \approx 0.8\frac{\omega_{0X}}{\gamma_X+\gamma_Y}$ for equal frequencies. Thus, one observes increasing $r_{X\to Y}$ with decreasing γ_X/ω_{0X} or γ_Y/ω_{0Y}.

The approximation (13) would be accurate for even smaller γ_X/ω_{0X} if one related the first (i.e., attained at smallest t) maximum of $F_{X\to Y}^2(t)$ to the equilibrium causal effect. However, the second difference of the case (ii) is that under decreasing damping terms the dependence $F_{X\to Y}^2(t)$ exhibits a progressively greater number of local maxima [Fig. 4(b)]. Its global maximum is attained at progressively greater $\tau_{X\to Y}$ which changes via jumps by about half an oscillation period [Fig. 5(e)]. Therefore, the maximal transient causal effect rises faster than the first maximum of $F_{X\to Y}^2(t)$ and the ratio $r_{X\to Y}$ starts to decrease with a further decrease of γ_X/ω_{0X} [at $\gamma_X/\omega_{0X} < 0.2$ in Fig. 4(e)]. In particular, it reaches the maximal value of $r_{X\to Y} \approx 2.6$ at $\gamma_X/\omega_{0X} \approx 0.1$. For smaller damping $\gamma_Y/\omega_{0Y} = 0.01$ (not shown), the ratio $r_{X\to Y}$ reaches a greater maximal value of 3.9 at a smaller $\gamma_X/\omega_{0X} = 0.05$. The value of $r_{X\to Y}$ for so narrow-band oscillators in the system (4) coincides with that for the relaxation systems (5) whose relaxation rates are $\alpha_X = \gamma_X$ and $\alpha_Y = \gamma_Y$, i.e., accounting only for the decaying envelopes of the oscillators' ACFs suffices to describe temporal dependences $F_{X\to Y}^2(t)$. It leads to a conjecture that for $\gamma_Y/\omega_{0Y} \to 0$ the ratio $r_{X\to Y}$ takes on progressively greater maximal values tending to 4.9 [maximal possible value of $r_{X\to Y}$ for the system (5) as shown in Ref. 16] attained at progressively smaller values of γ_X/ω_{0X}.

Finally, Fig. 6 shows the phenomenon of resonant behavior of the ratio $r_{X\to Y}$. The usual linear resonance in the system (4) with $\gamma_X/\omega_{0X} \ll 1$ and $\gamma_Y/\omega_{0Y} \ll 1$ manifests itself as a relatively narrow peak in the variance $\sigma_y^2(0, c_{YX})$ and, hence, in $S_{X\to Y}$ [Fig. 6(b), solid lines] under variation in the driving frequency ω_{0X} for all other parameters fixed, including the coupling coefficient (see caption to Fig. 6). However, the maximal transient effect $F_{X\to Y,\max}^2$ also exhibits a resonant behavior [Fig. 6(a), solid lines] so the question about $r_{X\to Y}$ is not clear in advance. Figure 6(c) shows that the resonance in $r_{X\to Y}$ is seen as clearly as that in $S_{X\to Y}$ for reasonably narrow-band oscillators (solid lines) and, moreover, a broad peak in $r_{X\to Y}$ is observed even for strongly damped oscillators where the resonances in $F_{X\to Y,\max}^2$ and $S_{X\to Y}$ are not seen (Fig. 6, dashed lines).

In summary, the oscillatory system (4) can exhibit the ratios $r_{X\to Y}$ both very small [less than those for the respective systems (6) and (7)] and quite large [close to the maximal one for the system (5)]. The case of equal frequencies in the system (4) provides the largest possible $r_{X\to Y}$ (13), while the smallest ratios are observed when a fast oscillator drives a slow one (11). Similarly, the maximum response time for the system (4) varies in a wider range and depends on the oscillators' parameters in a more complicated way than that for the respective systems (5), (6), and (7).

V. DISCUSSION

The equilibrium-to-transient causal effects ratio $r_{X\to Y}$ is found to vary in a wide range (Table I) from values arbitrarily close to zero (one subsystem is narrow-band, another is either wide-band or has a different frequency) up to about 5 (two narrow-band oscillators with equal frequencies). The formulas (8)–(13) relate the equilibrium causal effect $S_{X\to Y}$ to the maximal transient causal effect $F_{X\to Y,\max}^2$. Expressions for $S_{X\to Y}$ via the very short-term response rate and, hence, via $F_{X\to Y}^2(\Delta t)$ are also derived above. These formulas can be applied in practice to extract $S_{X\to Y}$ from time series (Sec. V C). Consideration of this issue is preceded by the discussion of limitations and possible extensions of the formulas (Sec. V A) as well as complications due to nonlinearity (Sec. V B).

A. Limitations and extensions

The first limitation consists in the condition of a sufficiently weak coupling which is necessary to obtain expressions for $r_{X\to Y}$ independent on the coupling coefficient c_{YX}. Previous numerical experiments[16] have shown that the formula (8) for the relaxation systems (5) gives the ratio $r_{X\to Y}$ with the relative error less than 10%, if $S_{X\to Y}$ is less than a certain upper bound depending on parameters: $S_{X\to Y} < 0.17$ for $\alpha_X/\alpha_Y = 1$, $S_{X\to Y} < 0.4$ for $\alpha_X/\alpha_Y = 10$, and $S_{X\to Y} < 0.1$ for $\alpha_X/\alpha_Y = 0.1$. Numerical results here show that the respective conditions for oscillatory systems are similar or even milder. If the relaxation system X in (5) is replaced with a strongly damped oscillator ($\alpha_X = \gamma_X$, $\gamma_X/\omega_{0X} \approx 1$), upper bound values of $S_{X\to Y}$ assuring less than 10% error of the approximation (9) are close to those for the approximation (8) at the respective α_Y.

If the damping in X is smaller, e.g., $\gamma_X/\omega_{0X} = 0.05$, the situation differs. For $\alpha_Y/\omega_{0X} = 0.02$, the upper bound $S_{X\to Y}$ is about 0.25, and for $\alpha_Y/\omega_{0X} = 10$ it is about 0.3. For $\alpha_Y/\omega_{0X} = 1$ the error of the approximation (9) remains less than 7% even for $S_{X\to Y}$ up to unity. If the relaxation system Y in (5) or (6) is replaced with an oscillator, the upper bounds for applicability of the respective formulas (10)–(13) remain almost unchanged. Thus, the obtained formulas for $r_{X\to Y}$ remain valid within the error of 10% up to considerable values of $S_{X\to Y}$ (typically, from 0.2 to 0.4), the strictest condition $S_{X\to Y} < 0.1$ corresponds to a relaxation system driving a much faster one. If necessary, for greater values of $S_{X\to Y}$ the ratio $r_{X\to Y}$ can be corrected to higher values maintaining the validity of the approximating formulas. However, such cases of strong couplings are often not the most interesting in practice, so the limitation on coupling strength is not very restrictive.

Another limitation is related to possibly bidirectional couplings. The equilibrium causal effect is then determined by the coupling coefficients in both directions, while the transient causal effects are determined mainly by the respective coupling coefficient if the coupling strength is not too large. After zeroing one of the coupling coefficients, one gets unidirectionally coupled systems and can introduce "unidirectional" causal effect for the respective direction. In the previous study of relaxation systems,[16] the equilibrium causal effect (3) has appeared to be a function of such elemental unidirectional

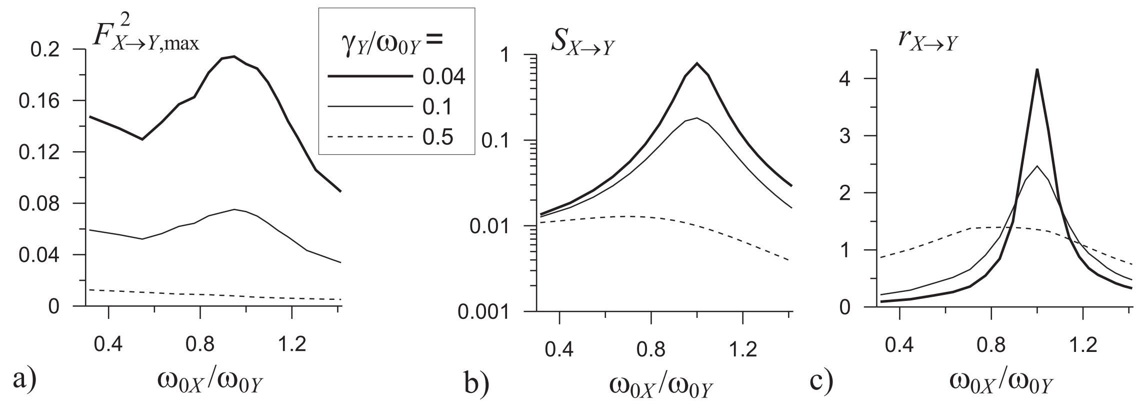

FIG. 6. Causal effects in the system of coupled oscillators (4) versus the ratio of frequencies for $\omega_{0Y} = 1$, $c_{YX} = 0.1$, $\sigma^2_{0x} = \sigma^2_{0y} = 1$, $\gamma_X/\omega_{0Y} = 0.1$, and different damping factors of the driven oscillator (the legend): (a) the maximal transient causal effects; (b) the equilibrium causal effects; (c) the ratio $r_{X\to Y}$ of those effects. Resonant character is exhibited by all plots if the damping is weak enough.

effects and some other system parameters. A search for such a function in case of oscillatory systems (4) may deserve further study and the two kinds of equilibrium causal effects (full and unidirectional) seem to be a relevant extension of the concept.

A very restrictive condition is the form of coupling function. All the formulas are obtained here for the simplest (but quite general as a first approximation) linear couplings in Eqs. (4)–(7). If coupling is parameterized in another way (e.g., if it is diffusive), expressions for the ratio $r_{X\to Y}$ may well appear different. This is an inevitable property of the equilibrium causal effect which compares dynamics at two values of coupling coefficient and, therefore, depends on the coupling parameterization by definition. Since the coupling considered here is elemental, the obtained formulas may well be useful to be generalized to other, broader classes of coupling functions.

B. Nonlinear oscillators

Nonlinearity of individual oscillators (4) can often violate the obtained relationships, but can also leave them unchanged. As an example of the latter situation let the restoring force term $\omega_{0Y}y$ in (4) or (7) be replaced with $\omega_{0Y}y(a + by^2)$. The oscillator Y becomes nonlinear with "hard spring" nonlinearity for $a = 1, b > 0$ (oscillation period decreases with amplitude) and "soft spring" for $a = 1, b < 0$. The equilibrium causal effect can be shown to decrease (increase) with $|b|$ in the former (latter) case since the restoring force rises (decreases) with amplitude. The transient causal effect is expected to depend on b in approximately the same way for the same reason. If so, the ratio $r_{X\to Y}$ should quite weakly depend on b if the latter is not too large ($|b| < 1/\langle y^2\rangle$ should hold in any case). Accordingly, the formulas (9)–(13) for $r_{X\to Y}$ should also work well in a wide enough range of $|b|$. Obviously, they remain quite accurate if $|b|$ is sufficiently small.

As an example of inapplicability of the obtained formulas, consider a double-well ($a = -1, b = 1$) overdamped ($\gamma_Y \gg \omega_{0Y}$) oscillator Y. This system demonstrates random jumps between two states "0" (around $y = -1$) and "1" (around $y = 1$) with frequency rising with the noise intensity $\Gamma_{\xi,Y}$. A simplified model of this oscillator useful to obtain exact results is a discrete-time two-state Markov chain with transition probabilities q_{Y0} (from state "0" to state "1") and q_{Y1} (from state "1" to state "0"). The values $q_{Y0} \neq q_{Y1}$ correspond to asymmetric potential of the original oscillator or different noise intensities in the two wells. For definiteness, let the system X be a similar two-state Markov chain with parameters q_{X0} and q_{X1}. Then, an influence $X \to Y$ is expressed via changes in the transition probabilities of Y depending on the current state of X. Let $q_{Y0} + \Delta q_{Y0|1}$ be the probability for Y to jump from "0" to "1" under the condition that the current state of X is "1," and $q_{Y0} + \Delta q_{Y0|0}$ under the condition that it is "0." Similarly, the probabilities to jump from "1" to "0" read $q_{Y1} + \Delta q_{Y1|1}$ and $q_{Y1} + \Delta q_{Y1|0}$. The quantities Δq_Y are coupling strengths. Numerical experiments show that quite different values of $r_{X\to Y}$ can be encountered.

First, it may be that the equilibrium distribution of Y does not change with the coupling strength while transient causal effects rise strongly and even tend to infinity, i.e., $r_{X\to Y} = 0$ for arbitrarily strong couplings. This happens for $q_{Y0} = q_{Y1} = q_{X0} = q_{X1}$ and a "symmetric influence," i.e., if the "0" state of X increases the probability for Y to remain at "0" and to quit "1" by the same value of $\Delta q_{Y1|0}$, and the "1" state of X changes by the same value the probability for Y to remain at "1" and to quit "0."

Second, the equilibrium causal effect may be negative, i.e., $r_{X\to Y} < 0$. It occurs for $q_{X0} \neq q_{X1}$ and symmetric coupling since the latter decreases the equilibrium variance of Y. It also occurs if $q_{X0} = q_{X1}$ and coupling is asymmetric: $\Delta q_{Y1|0} = -\Delta q_{Y0|0} > 0$ and $\Delta q_{Y0|1} = \Delta q_{Y1|1} = 0$.

Third, $r_{X\to Y}$ may well be positive as in the linear cases, but take on much greater values. It happens if $q_{Y0} \neq q_{Y1}$ and coupling tends to make these probabilities closer to each other, e.g., the ratio $r_{X\to Y}$ ranges almost up to 100 for $q_{X1} = q_{Y0} = 0.1, q_{Y1} = q_{X0} = 0.5$, and symmetric coupling, i.e., it exceeds the maximal values of $r_{X\to Y}$ obtained for the linear systems above by more than an order of magnitude.

To summarize, nonlinear systems are much richer in terms of possible relationships between transient and equilibrium causal effects. Further study of the latter for some basic classes of nonlinear oscillators seems quite relevant. However, sufficiently weak nonlinearity allows one to apply the obtained formulas (8)–(13) at least to get a first approximation.

C. Application procedure

The relationships (8)–(13) can be applied in practice to extract the equilibrium causal effect $S_{X\to Y}$ from a time series of coupled oscillators X and Y. In doing so, one assumes that one of the model equations (4)–(7) is valid, i.e., the limitations discussed above are not met. The latter can be learnt from *a priori* information and plots of the sample ACFs. Using the latter, one can also estimate relaxation times and oscillation periods of both systems assuming that the coupling is not too strong, so the ACF is not distorted as compared to the uncoupled case.

As mentioned above, the transient causal effects at various times can be estimated using normalized prediction improvements (the Wiener–Granger causality measure) at respective prediction times. At small prediction times, they provide short-term transient causal effects $F^2_{X\to Y}(\Delta t)$. Being maximized over prediction times, they give the maximal transient causal effect $F^2_{X\to Y,\max}$. Next, one can use either the estimate of $F^2_{X\to Y,\max}$ and the approximate formulas (8)–(13) or the estimate of $F^2_{X\to Y}(\Delta t)$ and the respective expressions to obtain the value of $S_{X\to Y}$. Ideally, both versions should give coinciding results which can be a further test confirming the validity of the model assumed. If the two estimates disagree strongly, then the validity of the model assumptions is questioned. In particular, if the dynamics over small times is dominated by noise, so the maximal transient causal effect is estimated more reliably, the latter should be preferred for estimation of $S_{X\to Y}$.

Details of the application procedure deserve a careful study. In particular, selection of state variables[65] for an analysis may well appear appropriate. Here, I only mention that first attempts of applying the formulas obtained here to real-world time series already gave meaningful improvements of the previous results[16,66] concerning the following climate example. A unidirectional influence of equatorial Atlantic mode (its monthly index X represents sea surface temperature in the equatorial Atlantic Ocean) on the El-Nino/Southern Oscillation (ENSO, its monthly index Y represents sea surface temperature in the eastern equatorial Pacific Ocean) was found in several works.[67–69] This influence was estimated[16,66] to determine 12% of the ENSO index variance assuming the validity of the simplest model (5). However, ACF of the ENSO index exhibits signs of the strongly damped oscillator with $\gamma_Y/\omega_{0Y} \approx 0.5$, so the model (7) should be more relevant. The corresponding formula (10) gives $S_{X\to Y} \approx 0.25$, i.e., about twice as large, which is a considerable difference. More detail on this and other climate examples will be reported elsewhere.

VI. CONCLUSIONS

Possible numerical values of the equilibrium-to-transient causal effects ratio $r_{X\to Y}$ are studied for unidirectionally coupled stochastic linear oscillators, including the cases of weakly and strongly damped oscillators, under the condition that the coupling is linear and not too strong. This ratio can approach an upper bound of approximately 5 for (i) two overdamped oscillators when relaxation time of the driver is much smaller than that of the response and (ii) two quite weakly damped (i.e., very narrow-band) oscillators with very close peak frequencies. This ratio can be arbitrarily small: if one of the oscillators is weakly damped and the other one is either strongly damped or possesses a different peak frequency, then $r_{X\to Y}$ is of the order of the ratio "damping factor-to-natural frequency" for the weakly damped oscillator, its values down to 0.03 are possible if the oscillators' time scales differ by no more than an order of magnitude. This range differs from $1 \le r_{X\to Y} \le 5$ reported previously[16] for overdamped oscillators. Dependencies of both causal effects and $r_{X\to Y}$ on parameters of the oscillators are shown, including resonance in $r_{X\to Y}$ for narrow-band oscillators.

Simple approximate formulas (8)–(13) for the ratio $r_{X\to Y}$ depending on oscillation periods and relaxation times are obtained. They can be used to extract equilibrium causal effects from a time series recorded at a fixed coupling strength. Such applications can be performed using estimates of transient causal effects provided by the well-established Wiener–Granger causality. Being impossible at the first glance, estimation of equilibrium causal effects appears feasible for well defined (and sufficiently broad) class of oscillatory systems considered here. Limitations and possible generalizations of the obtained relationships are discussed in terms of the strength, structure (e.g., uni- or bidirectionality), and functional form (parameterization) of coupling and nonlinearity of the oscillators. The latter two circumstances impose the most considerable restrictions and seem to deserve further studies for selected classes of nonlinearities and coupling parameterizations.

Finally, it can be noted that such a popular and widely used coupling characteristic as transfer entropy[18] is an information-theoretic version of the Wiener–Granger causality and for linear systems it is equivalent[59] to the prediction improvement in terms of mean squared errors used here. For nonlinear systems, transfer entropy and such prediction improvement are different quantitative characteristics of transient causal effects. As well, for nonlinear systems one may well be interested in other forms of equilibrium causal effects, e.g., characterizing equilibrium probability distributions in the information-theoretic terms. Relationships between such characteristics and transfer entropy for nonlinear systems may well appear more universal than the ratio studied in this work.

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APPENDIX: COMPUTATIONAL TECHNIQUE

The technique used to determine both equilibrium and transient causal effects is briefly described here; more details are given in Refs. 10 and 16. A linear stochastic system used here as a basic object can be written in the form $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \xi$, where $\mathbf{z}$ is a d-dimensional state vector consisting of the components $\mathbf{x}$ and $\mathbf{y}$, matrix $\mathbf{A}$ specifies both individual properties of X and Y and coupling, $\boldsymbol{\xi}$ is d-dimensional Gaussian white noise with zero mean, and ACF $\langle \xi(t_1)\xi^{\mathrm{T}}(t_2)\rangle = \Gamma\delta(t_1 - t_2)$, T standing for transposition. For an initial state $\mathbf{z}(0) = \mathbf{z}_0$ and any $t > 0$, the conditional distribution $\rho_t(\mathbf{z}(t)\,|\mathbf{z}(0) = \mathbf{z}_0)$ is Gaussian with expectation $\mathbf{m_z}(t)$ and covariance matrix $\mathbf{C_{zz}}(t)$. The latter two quantities can be found via solving linear ordinary differential equations $\dot{\mathbf{m}}_\mathbf{z} = \mathbf{A}\mathbf{m}_\mathbf{z}$ and $\dot{\mathbf{C}}_\mathbf{zz} = \mathbf{A}\mathbf{C}_\mathbf{zz} + \mathbf{C}_\mathbf{zz}\mathbf{A}^\mathbf{T} + \Gamma$. Having these conditional moments, one can find the transient causal effects based on the definition (2) in a straightforward manner.[10] The equilibrium causal effect is found after finding the stationary (equilibrium) variances via solving the linear algebraic equation $\mathbf{A}\mathbf{C}_\mathbf{zz} + \mathbf{C}_\mathbf{zz}\mathbf{A}^\mathbf{T} = -\Gamma$.

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