Chaos, Solitons and Fractals 148 (2021) 110972

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Class-oriented techniques for reconstruction of dynamics from time series

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ARTICLE INFO

Article history:
Received 26 March 2021
Accepted 11 April 2021

Keywords:
Reconstruction of dynamical systems
Time series analysis
Parameter estimation
Hidden variables
Time-delay systems
Phase oscillators
Directional couplings

ABSTRACT

Reconstruction of dynamical systems from time series is an important problem intensively studied within nonlinear dynamics and time series analysis for the last three decades. Its solution is a tool to accomplish prediction, classification, diagnostics and many other tasks. Universal approaches are quite attractive, but more specific techniques based on prior information about a system under study often appear advantageous in practice. We present an overview of the works of our team where such "class-oriented" techniques have been developed for realistic situations differing by the degree of prior knowledge: fully known structure of the dynamics equations with an accent to dealing with hidden variables and partly known structure for time-delay systems and coupled phase oscillators.

1. Introduction

Studying living systems is a mainstream of contemporary science. It has got this position to a good deal due to successful development of technology, drastic improvement of available means for information retrieval and storage, achievements in computer science and nonlinear dynamics. These factors provided realization of novel methods for experimenting with living systems and for empirical modeling, in particular, for reconstruction of mathematical models from time series of observed variables. The latter is quite timely since possibilities of living systems modeling from first principles are limited both due to complexity of those objects and lack of such universal laws as Newton's or Maxwell's equations in the field of living systems. Professor Vadim Anishchenko in the last decade of XX century paid careful attention to the problems of reconstruction of nonlinear models from time series and published a number of works concerning these problems [1-8]. As his "neighbors" in the university and research (and some of us also as his students), we present here our contribution to this topical field.

According to the degree of prior information about a system under study, one can single out [9,10] the problems of "black box" (no information), "white" or "transparent box" (fully known form of equations, only parameter values unknown), and "grey box" (partial knowledge of possible model structure). Along with many other authors (e.g. [11-14]), we have dealt with reconstruction problems under each of these settings in our previous works, e.g., addressing the choice of dynamical variables for a black box situation [15], forming a specific model structure for a grey box case of regularly driven systems [16], and developing parameter estimation technique for one-dimensional chaotic maps as a white box situation [17]. Overall, it appears that success of modeling from data series is often determined by specialization, when a modeling technique is developed for sufficiently narrow classes of objects and takes into account their specific features. In this paper, we overview several practically important specific situations (classes) and the respective "class-oriented" techniques developed in the previous works of our team over the last two decades. These situations include parameter estimation in case of hidden variables under fully known model structure (Section 2), reconstruction of delay times in the systems governed by delay-differential equations (Section 3), revealing couplings and estimation of coupling delays from phase dynamics (Section 4). We conclude in Section 5.

2. Hidden variables and parameter estimation

Multiple methods for reconstruction of equations from time series also assume that some components of state vector cannot be measured. When the model structure is unknown, Takens' theorem [18] together with false nearest neighbor techniques [19] justify estimation of the dimension of a reconstructed state vector. Sequential differentiation [20] or time delay embedding [21,22] are

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https://doi.org/10.1016/j.chaos.2021.110972
0960-0779/

then used to get the state vectors. These techniques are often used if these is no prior knowledge about model equations form and physical meaning of their parameters [23-26]. When the "white box" problem is considered, physical meaning of all state variables and parameters is indicated. This makes time delay embedding and differentiation useless except some special cases like van der Pol like generators [27] or specific phase-locked loops [28] in which all unobserved variables are the derivatives or integrals of an observable. As for unobserved variables and unknown parameters entering model equations, they are usually of great interest for a researcher and the "white box" modeling serves as a tool for their "indirect measurement" [10]. The hidden variable problem is right the problem of recovering unknown parameters and state variables which cannot be directly measured or computed from an observed time series in case of fully known structure of model equations. It might seem to be the simplest reconstruction task, but appears quite difficult in practice.

2.1. Initial value (single shooting) approach

The earliest implementation of hidden variable approach was proposed in [29]. The problem is following. We have a model (2.1) in the form of $\vec{x} = (x_1, \dots, x_D)$ ODEs (the state vector is $\vec{x} = (x_1, \dots, x_D)$) with M free (i.e. to be estimated) parameters $\vec{c} = (c_1, c_2, \dots, c_M)$:

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, \vec{c}) \tag{2.1}$$

The functions $\vec{f} = (f_1, f_2, \dots, f_D)$ are known, but only part of the state vector $\vec{x}$ components are observed in the form of a vector time series $x_i(t_n), i = 1, \dots, (D-Q)$ of the length $(N+1)$ at time instants $(t_0, t_1, \dots, t_N)$, other Q variables are hidden. Those hidden variables cannot be obtained using any direct technique like time delay embedding or sequential differentiation. Therefore, estimation of $\vec{c}$ from time series via usual regression is impossible.

The idea of hidden variable approaches is to consider the initial conditions for hidden variables as additional unknown parameters. One makes a starting guess for the parameter vector $\vec{c}$ (denote it $\tilde{\vec{c}}$) and for the hidden variables initial values $(s_{Q+1}(t_0), \dots, s_D(t_0))$. For that starting guess, one computes a trajectory of the model by solving the ODEs numerically. The obtained model time series of the variables $(x_1, \dots, x_Q)$ are compared to their originally observed time series. Now, combine parameters and initial conditions into a single vector $\zeta_i, i = 1, \dots, K = M + Q$, where the first M components are parameters and the last Q ones are initial conditions for the hidden variables. One constructs a cost function $S(\vec{\zeta})$ as a weighted sum of square differences (2.2) between the originally observed $x_i(t_n)$ and the model $\tilde{x}_i(t_n)$ values of the first $(D-Q)$ variables.

$$S(\vec{\zeta}) = \sum_{i=Q+1}^{D} w_i \sum_{n=1}^{N} (\tilde{x}_i(t_n) - x_i(t_n))^2 \tag{2.2}$$

The weights w_i can be quite generally set as $w_i = 1/\sigma_i^2$ where σ_i is the standard deviation of the i-th variable which is readily estimated from data. Then, one minimizes the function S. This function is implicit and its derivatives can be calculated only numerically. As an option, consider the gradient descent method. It assumes that a correction $\overrightarrow{\Delta\zeta}$ to a current guess $\vec{\zeta}$ is made via solving Eq. (2.3), where $\hat{H}(\vec{\zeta})$ and $\vec{g}(\vec{\zeta})$ are Hessian and gradient of S at $\vec{\zeta}$, respectively:

$$\hat{H}(\vec{\zeta})\overrightarrow{\Delta\zeta} = -\vec{g}(\vec{\zeta}) \tag{2.3}$$

To compute $\hat{H}(\vec{\zeta})$ and $\vec{g}(\vec{\zeta})$ one has to perturb the trajectory via changing each component of $\vec{\zeta}$ in turn by a small value $\delta\zeta_i$ (in general, it is taken to differ for different components) and solving Eq. (2.1) to obtain the trajectories corresponding to the perturbed $\vec{\zeta}$. For example, i-th component of the gradient can be estimated as follows:

$$g_i = \frac{\partial S}{\partial \zeta_i} \approx \frac{1}{2\delta\zeta_i}(S(\zeta_1, \dots, \zeta_i + \delta\zeta_i, \dots, \zeta_K) - S(\zeta_1, \dots, \zeta_i - \delta\zeta_i, \dots, \zeta_K)). \tag{2.4}$$

This requires solving the equations $2K$ times. Analogously, a component $h_{i,j}$ of the Hessian $\hat{H}$ can be estimated using the formula (2.5) which requires solving Eq. (2.1) $4K^2$ times.

$$\begin{aligned} h_{i,j} &= \frac{\partial^2 S}{\partial\zeta_i \partial\zeta_j} \approx \frac{S_{+i+j} - S_{+i-j} - S_{-i+j} + S_{-i-j}}{4\delta\zeta_i\delta\zeta_j} \\ S_{+i+j} &= S(\zeta_1, \dots, \zeta_i + \delta\zeta_i, \dots, \zeta_j + \delta\zeta_j, \dots, \zeta_K), \\ S_{+i-j} &= S(\zeta_1, \dots, \zeta_i + \delta\zeta_i, \dots, \zeta_j - \delta\zeta_j, \dots, \zeta_K), \\ S_{-i+j} &= S(\zeta_1, \dots, \zeta_i - \delta\zeta_i, \dots, \zeta_j + \delta\zeta_j, \dots, \zeta_K), \\ S_{-i-j} &= S(\zeta_1, \dots, \zeta_i - \delta\zeta_i, \dots, \zeta_j - \delta\zeta_j, \dots, \zeta_K). \end{aligned} \tag{2.5}$$

The formulae (2.3-2.5) provide an iterative algorithm to obtain estimates for all initial conditions and parameters. If the step size appears large, the value $S(\vec{\zeta} + \overrightarrow{\delta\zeta})$ at some step can become larger than $S(\vec{\zeta})$. Then, a reduced step $\overrightarrow{\delta\zeta}/2$ is to be applied. If $S(\vec{\zeta} + \overrightarrow{\delta\zeta}/2) \geq S(\vec{\zeta})$, the $\overrightarrow{\delta\zeta}/4$ is considered and so on, until either the condition $S(\vec{\zeta} + \overrightarrow{\delta\zeta}/2^i) < S(\vec{\zeta})$ would be satisfied on i-th iteration, or a preset maximum number of iterations i_{max} is reached. In the last case the algorithm is stopped and an achieved local extremum is considered as its result.

2.2. Multiple shooting approach and its extensions

In case of chaotic time series, the main problem of the initial value approach is that a long enough model trajectory $\tilde{\vec{x}}(t)$ is very sensitive to small perturbations of a current guess due to positive Lyapunov exponent. Actually, if the observed series length is about 5 times as large as the Lyapunov time (an exact "threshold" value depends on the measurement noise and other factors), the reconstruction becomes practically impossible since the basin of attraction of global minimum becomes too narrow.

A possible solution was proposed by Bock et al in [29-31]. The idea is to consider multiple initial conditions distributed along the trajectory: $\tilde{\vec{x}}(t_0), \tilde{\vec{x}}(t_\eta), \tilde{\vec{x}}(t_{2\eta}), \dots, \tilde{\vec{x}}(t_{(L-1)\eta})$, where η is the number of points in one piece of the time series and L is a number of parts, so that $N = \eta \times L$. If so, the calculation formula for the cost function (2.2) remains the same, but dimension of $\vec{\zeta}$ is larger:

$$\vec{\zeta} = \left(c_1, \dots, c_M, \tilde{x}_{Q+1}(t_0), \dots, \tilde{x}_{Q+L}(t_0), \tilde{x}_{Q+1}(t_\eta), \dots, \tilde{x}_{Q+L}(t_\eta), \dots, \tilde{x}_{Q+1}(t_{\eta(l-1)}), \dots, \tilde{x}_{Q+L}(t_{\eta(l-1)})\right) \tag{2.6}$$

An additional continuity restriction (2.7) is formulated in [29], which means that a trajectory starting from the condition $\vec{s}(t_{\eta-1})$ comes to the next one $\vec{s}(t_\eta)$:

$$\vec{x}\left(t_{i\eta}, \vec{s}_{\eta-1}\right) = \vec{s}_\eta\left(t_{i\eta}\right) \tag{2.7}$$

This leads to the conditional optimization problem. First, all L pieces of the time series are considered independently, with calculation of corrections similarly to the initial value approach. Then, a correction to initial conditions for all $\vec{s}_i$ except the last one is performed by means of linear decomposition of (2.7) and backward propagation of corrections.

This approach is more efficient. It was demonstrated that 3D system of equations such as Lorenz or Rössler's system can be reconstructed from a scalar time series of one of its variables, with 2-3 parameters estimated with a fine precision [29,32]. However, it appears to be still insufficient for many practical tasks. The problem of bad divergence for long chaotic time series persists, though the multiple shooting approach allows one to use longer time series than the initial value approach. A segmentation technique, when the entire original series is divided into a number of segments is not sufficient in most cases since if the estimates are biased for a single segment, their average would be also biased. To fix this problem, a number of elaborated techniques were proposed [32-36].

A modified Bock's technique. We focus here on a relatively simple extension of Bock's technique proposed in our work [32]. A similar idea was mentioned in [37]. At variance with a direct segmentation technique, the idea of [32] is to keep the same parameters for the whole series, but to allow ν discontinuities. This means that the condition (2.7) is not imposed at ν time instants. A resulting trajectory would be discontinuous that reduces the difficulty related to sensitivity of a chaotic trajectory to small perturbations.

In [32], a comparative study of the original Bock's technique and its above extension was performed for Lorenz (2.8) and Rössler's (2.9) systems. Since the results of both techniques significantly depend on the starting guesses of parameters, the calculations were performed for a large range of starting guesses in order to estimate the convergence radius r_μ in the normalized parameter space $(b_{1,\ldots,}b_M)$, where $b_i = (\tilde{c}_i - c_i)/c_i$ are the normalized to the actual (true) value parameters and $0 \le \mu \le 1$ is a ratio of the number of starting guesses for parameters from which the global minimum was reached to the whole number of trial starting guesses for parameters.

$$\begin{aligned}
\dot{x}_1 &= c_1 x_2 - x_1 \\
\dot{x}_2 &= -x_2 + x_1(c_3 - x_3) \\
\dot{x}_3 &= -c_2 x_3 + x_1 x_3 \\
\dot{x}_1 &= -x_2 - x_3 \\
\dot{x}_2 &= x_1 + c_1 x_2 \\
\dot{x}_3 &= c_2 + x_3(x_1 - c_1)
\end{aligned} \tag{2.9}$$

The convergence radius r_μ for different r for Lorenz and Rössler systems shows that the modified technique is approximately 1.2-1.4 times better than the original one, and in both cases, it is possible to get rather good estimates.

2.3. Delay-differential equation reconstruction

As it was previously mentioned, the larger is the number of unknown initial conditions to be reconstructed, the less are the chances for success due to the larger dimension K of the vector $\vec{\zeta}$. This is since the optimization problem complexity is rising very fast with increase of number of optimized function variables, including rise in number of local minima and reduction of global minimum basin of attraction. If we consider the time-delay equations, which are usually considered as a next step model after ODEs (a system that is somewhat more complex and regime rich but not completely different [38,39]), we face a great problem: the number of initial conditions to be set as unknown parameters is infinite and nothing can be done using hidden variable approach.

However, if we consider some numerical scheme for some particular situation, it seems to be a little bit different. If the system to be reconstructed from data has a small delay time, for which numerical scheme used to solve it is stable for relatively large time step (this actually means that the delay τ is "small in data points", i.e. $\tau = \theta\Delta t$, where $\theta \lesssim 10$), we can generally try to solve such a problem.

The first attempt to do this was performed in [40] for two unidirectionally coupled simplest DDEs with quadratic nonlinearity (2.10):

$$\begin{aligned}
\dot{x}(t) &= -x(t) + \lambda_x - x^2(t-\tau) + ky(t), \\
\dot{y}(t) &= -y(t) + \lambda_y - y^2(t-\tau),
\end{aligned} \tag{2.10}$$

where there are three unknown parameters λ_x, λ_y and k. The variable x of the driven oscillator was considered as an observable and the variable y of the driving one was hidden. Different regimes were tested. For some simple regular regimes with $\theta = 10$ or $\theta = 11$, the reconstruction was successful providing estimates of two parameters (λ_x and λ_y with k considered to be known) and times series of variable y. The results of parameter estimation for different starting guesses are shown in Fig. 2.2 in the same way as for Lorenz and Rössler's systems previously (b_1 and b_2 correspond to $\tilde{\lambda}_x$ and $\tilde{\lambda}_y$ for the fixed $\tilde{k} = k$). One has to notice that the parameter plane in Fig. 2.2 is very crossed compared to Fig 2.1 and even for starting guesses very close to the actual values there is a possibility not to reach the global minimum. Therefore, the radius $r_1 \approx 0$. This is a result of large number of initial conditions for the delayed hidden variable. For more complex regimes with larger θ, one can find neither global minimum nor any local one close to it.

Significant advance in the field is possible if one takes into account that the starting guesses for the hidden variable are not actually independent, but the realization has to be continuous. In [41] it was proposed to take into account only a small number L_s of them and to obtain the others by means of cubic spline interpolation. Though interpolation introduces some additional source of error, the results were significantly better than previously and it occurred to be possible to perform reconstruction for the 3D system (Lang-Kobayashi Eq.s (2.11)) with two unknown parameters from a single observable including the case of additive measurement noise:

$$\begin{aligned}
\dot{\rho}(t) &= F(t)\rho(t) + A\rho(t-\tau)\cos(\phi(t) - \phi(t-\tau) + \Omega\tau), \\
\rho(t)\dot{\phi}(t) &= \alpha F(t)\rho(t) \\
&\quad + A\rho(t-\tau)\sin(\phi(t) - \phi(t-\tau) + \Omega\tau), \\
T\dot{F}(t) &= P - F(t) - (1 + F(t))\rho^2(t),
\end{aligned} \tag{2.11}$$

where $\rho(t)$ and $\phi(t)$ are the modulus and the phase of complex electric field $E(t) = \rho(t)\exp(i\phi(t))$, respectively, $F(t)$ is the excess carrier number, T is the ratio of the carrier lifetime to the photon lifetime, P is the dimensionless pumping current above threshold, τ is the ratio of the external cavity round-trip time and the photon lifetime, A is the strength of the feedback, α is the linewidth enhancement factor, and Ω is the dimensionless angular frequency of the solitary laser. The parameters P and A are assumed to be unknown, while the variables $\phi(t)$ and $F(t)$ are hidden (see [41] for details).

Due to spline approximation, there is no any strict dependency on θ and it can be up to 2000 and even more. The only significant point is how many oscillations take place within the delay time. If additional smaller time scale is present as for the intermittency regime in Fig 2.3c, the results are much worse, while the global minimum is still reachable from at least 10% of the considered starting guesses. For both regular and chaotic regimes considered in Fig. 2.3a,b, this percentage is about 60%. The regular regime takes advantage of zero Lyapunov exponent, and the chaotic one provides more information for the reconstruction (the Lyapunov time is less than the length of used time series and, therefore, trajectory divergence is not significant for the reconstruction technique).

To finish this subsection, note that the "white box" problem addressed using hidden variable approaches seems to be the simplest one over all reconstruction problems. However, in this field there is a little progress in the last three decades. It seems that the main reason is that there is no possibility to avoid the global optimization of function of large number of variables. In some cases, reconstruction techniques based on global optimization of such functions occurred to be successful, even with great number of variables up to several hundred [42], but that was due to specific features of nonlinear functions in the equations (sigmoids) which

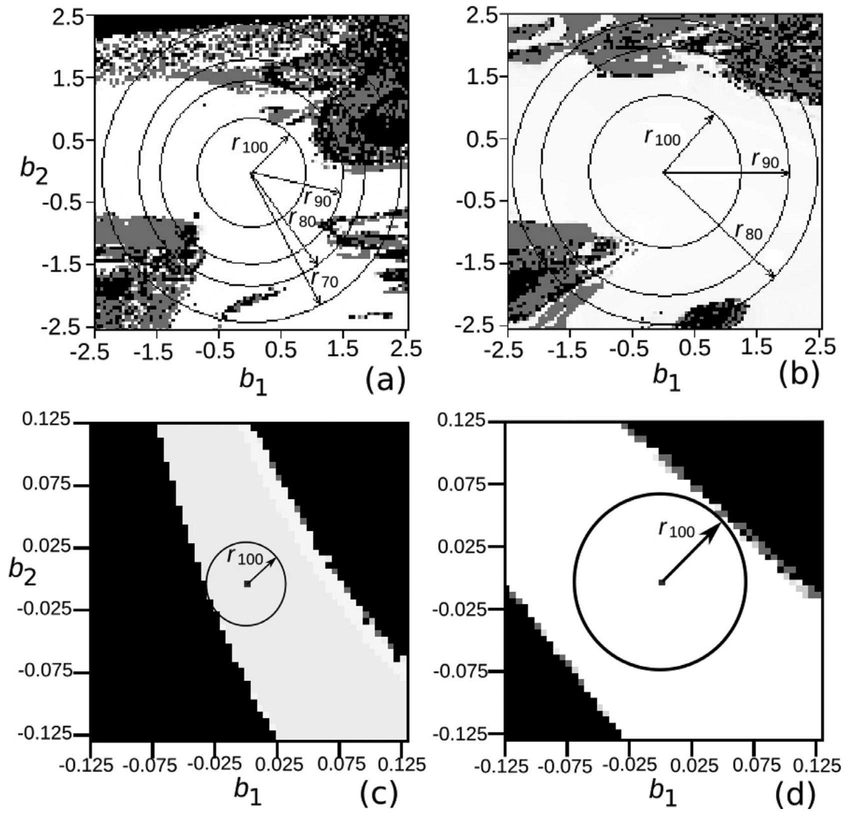

Fig. 2.1. "The map of convergence" — values of the cost function depending on two normalized parameters b_1 and b_2 of Lorenz system (2.8) — panels (a,b) and Rössler's system (2.9) — panels c,d are plotted in grey scale (the darker means the larger). The convergence radius r_μ for different values of μ.

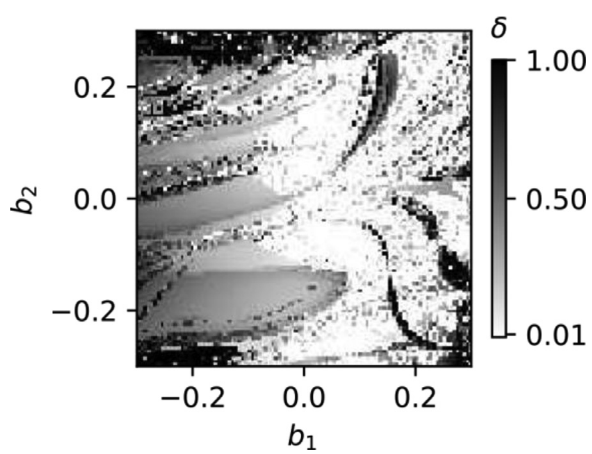

Fig. 2.2. "The map of convergence" — values of the target function depending on two normalized parameters b_1 and b_2 corresponding to $\tilde{\lambda}_x$ and $\tilde{\lambda}_y$ are plotted in grey scale.

circumstance protected the algorithm from divergence. Possibility to compute derivatives of the cost function explicitly (analytically) could be helpful too, but for the hidden variable problem setting the cost function itself is determined implicitly. There were attempts to escape the global optimization using machine learning techniques like reservoir computing [43] which can be considered as a kind of brute force. However, comparison of results obtained in [43] and in [41] definitely shows that specialized approaches occur to be more powerful, recovering more information in greater number of regimes with less additional assumptions. Another difficulty is that any inaccuracy in an evolution operator form or improper account of measurement function make hidden variable approaches inoperable since the cost function is constructed based on strict correspondence between the observed dynamics and modeling equations.

Nevertheless, advances in hidden variable technique development seem to be possible due to the following three opportunities. First, reduction of the number of initial conditions to be estimated and, therefore, simplification of the cost function is always helpful. Use of any additional information about the hidden variables like continuity, range and periodicity should help. Second, improved optimization techniques together with an increase in computational power suggest that some problems too complex now can be addressed in the future. Third, reformulation of the cost function based on a model generalization like it was done in [27,42,44,45] can reduce the method fragility.

3. Reconstruction of delay times from time series of delay-differential equations

Systems with time delays are widespread in nature, technology, and living systems. Many physical, chemical, climatic, and biological self-oscillating systems have time-delayed feedback, which has a great influence on their dynamics. Time delays must be taken into account when studying spatially developed systems in which signals propagate at a finite speed and they need time to cover distances. Generally, the time-delay systems are modeled by delay-differential equation

$$\varepsilon_n x^{(n)}(t) + \varepsilon_{n-1} x^{(n-1)}(t) + \cdots + \varepsilon_1 \dot{x}(t) = F(x(t), x(t-\tau_1), \ldots, x(t-\tau_k)), \tag{3.1}$$

where $x(t)$ is the system state at time t, $x^{(n)}(t)$ is the time derivative of order n, $\tau_1, \ldots, \tau_k$ are the delay times, $\varepsilon_1, \ldots, \varepsilon_n$ are the parameters characterizing the inertial properties of the system, and F is a function. System (3.1) has an infinite-dimensional phase space,

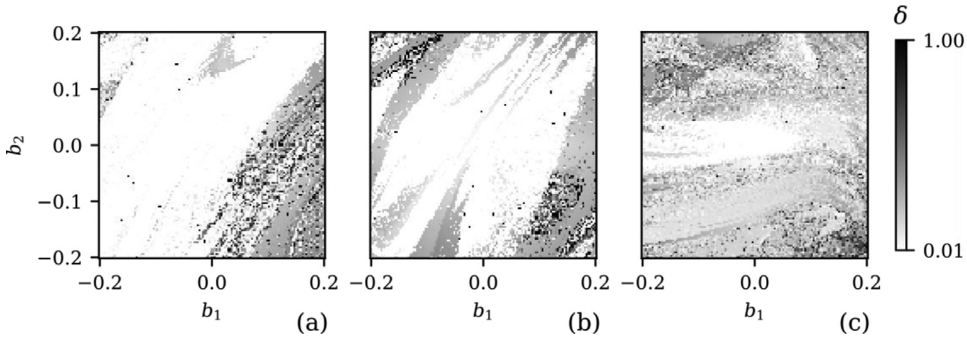

Fig. 2.3. "The map of convergence" — values of the cost function depending on two normalized parameters b_1 and b_2 corresponding to parameters A and P of the system (2.11) are plotted in grey scale (the darker means the larger) for three considered regimes: regular (a) with $\tau = 60.5$, chaotic (b) with $\tau = 63.5$, and intermittency (c) with $\tau = 65$.

because it is necessary to prescribe the initial conditions in the entire time interval $[-\tau_k,0]$ in order to uniquely define its dynamics. Therefore, one has to use special approaches for reconstructing time-delay systems, since their direct reconstruction using conventional time-delay embedding techniques often fails.

A lot of special methods have been proposed for recovering the parameters of time-delay systems and especially the delay times. Most of these methods are intended for the reconstruction of first-order systems (3.1) having a single delay time τ_1:

$$\dot{x}(t) = F(x(t), x(t-\tau_1)). \tag{3.2}$$

An efficient approach for reconstructing time-delay systems (3.2) is based on the projection of the system trajectory from the infinite-dimensional phase space to several three-dimensional spaces $(x(t-\tau), x(t), \dot{x}(t))$ upon variation of τ. The time derivatives $\dot{x}(t)$ in this case are numerically estimated from time series. If $\tau = \tau_1$, the projected trajectory is confined to a two-dimensional manifold, defined by $\dot{x}(t) - F(x(t), x(t-\tau_1)) = 0$. This property was used in [46-59] to identify time-delay systems (3.2) and to recover the delay time τ_1 and the function F. The mentioned papers employed different criteria of quality for the reconstructed equations, for example, various measures of complexity of the projected time series [46-50], measures based on the regression analysis [51-53], the minimal forecast error of constructed model [54–58], or the minimal value of information entropy [59]. There are a number of other special methods for reconstructing time-delay systems that are based on multiple shooting approach [60], information-theory approaches [61,62], adaptive synchronization [63,64], optimization techniques [65,66], or other approaches [67–69].

In [70], we have proposed a simple method for reconstructing the delay time of time-delay systems (3.2), which is based on statistical analysis of time intervals between extrema in the system time series. We have shown that there are practically no extrema in $x(t)$ separated in time by the delay time τ_1. Actually, differentiation of Eq. (3.2) with respect to t gives

$$\ddot{x}(t) = \frac{\partial F(x(t), x(t-\tau_1))}{\partial x(t)}\dot{x}(t) + \frac{\partial F(x(t), x(t-\tau_1))}{\partial x(t-\tau_1)}\dot{x}(t-\tau_1). \tag{3.3}$$

In a typical case of quadratic extrema, $\dot{x}(t) = 0$ and $\ddot{x}(t) \neq 0$ at the extremal points. Therefore, it follows from Eq. (3.3) that derivatives $\dot{x}(t)$ and $\dot{x}(t-\tau_1)$ do not vanish simultaneously, i.e., if $\dot{x}(t) = 0$, then $\dot{x}(t-\tau_1) \neq 0$. Defining, for different values of τ, the number N of situations where the points of $x(t)$ separated in time by τ are both extremal, we can construct the $N(\tau)$ plot and recover the delay time τ_1 as the value at which the absolute minimum of $N(\tau)$ is observed [70].

Fig. 3.1 shows the method application to time series of the Ikeda equation [71]

$$\dot{x}(t) = -x(t) + \mu \sin(x(t-\tau_1) - x_0) \tag{3.4}$$

modeling the passive optical resonator system. The parameters of the system (3.4) are chosen to be $\mu = 20$, $\tau_1 = 2$, $x_0 = \frac{\pi}{3}$ to produce a dynamics on a high-dimensional chaotic attractor. Part of the time series is shown in Fig. 3.1(a). The time series is sampled in such a way that 200 points in time series cover a period of time equal to the delay time τ_1. The time series contains 20000 points and exhibits about 1100 extrema. For various τ values we count the number N of situations when $\dot{x}(t)$ and $\dot{x}(t-\tau)$ are simultaneously equal to zero and construct the $N(\tau)$ plot, Fig. 3.1(b). The step of τ variation in Fig. 3.1(b) is equal to the integration step $h = 0.01$. The absolute minimum of $N(\tau)$ takes place exactly at $\tau = \tau_1 = 2.00$.

The method turned out to be resistant to noise. The absolute minimum in the $N(\tau)$ plot becomes less pronounced in the presence of noise, but is still distinguished at moderate noise levels. For example, in the considered above case of Ikeda equation, the location of the minimum of $N(\tau)$ allowed us to estimate the delay time accurately even when a zero-mean Gaussian white noise added to the system time series had a standard deviation of 20% of the standard deviation of the data without noise (the signal-to-noise ratio was about 14 dB). The method can be extended to time-delay systems (3.1) of high order and with several coexisting delays [72]. It is still efficient for the recovery of delay times of coupled time-delayed feedback systems from their time series [73,74].

Another approach for reconstructing the delay time of delayed feedback systems is based on the nearest neighbor method, which is widely used in different scientific disciplines for classification of systems [75] and prediction of their time series [76]. In [77], we have proposed to employ the nearest neighbor method for the recovery of delay time from time series of time-delay systems. To explain the method idea, let us consider a time-delay system (3.2) of the following form:

$$\varepsilon_1 \dot{x}(t) = -x(t) + f(x(t-\tau_1)), \tag{3.5}$$

where f is a nonlinear function.

Analyzing time series, we always deal with variables measured at discrete instants of time. Therefore, we pass from differential Eq. (3.5) to the difference equation

$$\varepsilon_1 \frac{x(t+\Delta t) - x(t)}{\Delta t} = -x(t) + f(x(t-\tau_1)), \tag{3.6}$$

where Δt is the sampling time. Eq. (3.6) can be rewritten as

$$x(t+\Delta t) = a_1 x(t) + a_2 f(x(t-\tau_1)), \tag{3.7}$$

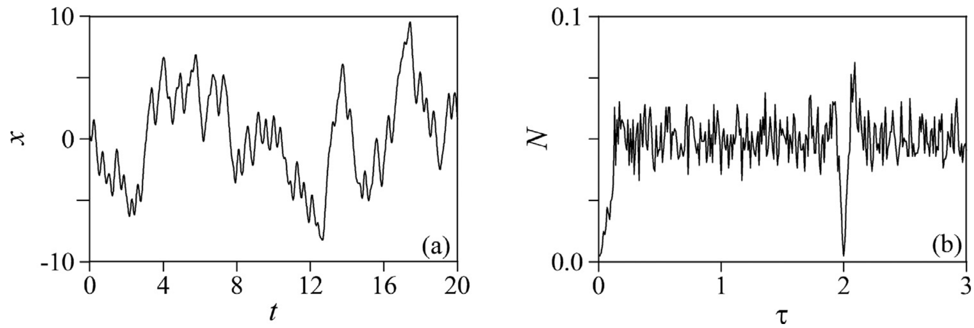

Fig. 3.1. (a) Chaotic time series of the Ikeda Eq. (3.4). (b) Number N of pairs of extrema in the time series separated in time by τ, as a function of τ. $N(\tau)$ is normalized to the total number of extrema in the time series.

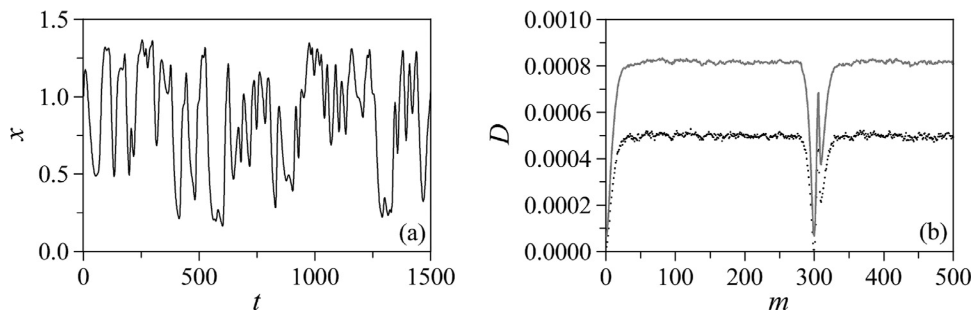

Fig. 3.2. (a) Chaotic time series of the Mackey-Glass Eq. (3.11). (b) Dependences of D on the trial delay time m for $k = 1$ (dotted black line) and $k = 10$ (solid grey line).

where $a_1 = 1 - \frac{\Delta t}{\varepsilon_1}$ and $a_2 = \frac{\Delta t}{\varepsilon_1}$. Let us write Eq. (3.7) in the form of the discrete-time map

$$x_{n+1} = a_1 x_n + a_2 f(x_{n-d}), \tag{3.8}$$

where $n = \frac{t}{\Delta t}$ is the discrete time and $d = \frac{\tau_1}{\Delta t}$ is the discrete delay time.

Assume that we have a time series $\{x_n\}_{n=1}^{N}$ of the system (3.5), where N is the number of points. We define vector $\vec{X}_i = (x_i, x_{i-d})$ and find its nearest neighbor $\vec{X}_j = (x_j, x_{j-d})$ using the Euclidean metrics

$$L(\vec{X}_i, \vec{X}_j) = \sqrt{(x_i - x_j)^2 + (x_{i-d} - x_{j-d})^2}, \tag{3.9}$$

where $j \neq i$. The vector $\vec{X}_j$ will be the nearest neighbor of $\vec{X}_i$, if the distance $L(\vec{X}_i, \vec{X}_j)$ is minimal. Generally, it is a common practice to find not one, but k nearest neighbors for a given vector. The nearest neighbor vectors containing the dynamical variable of the system (3.8) at the instants of time n and $n - d$, where $n \in [d + 1, N - 1]$, will lead to the close states of the system at the instants of time $n + 1$. Since the delay time is *a priori* unknown, we vary the trial delay times m within some interval and for k nearest neighbors of each vector $\vec{X}_n = (x_n, x_{n-m})$ constructed from the time series, estimate the variance σ_n^2 of the system states at the corresponding instants of time $n + 1$.

In the case of false choice of m ($m \neq d$), the variance of these states may be great, because the system states at the instants of time $n + 1$ do not depend on the system states at the instants of time $n - m$. True delay time d can be estimated as the value at which the dependence

$$D(m) = \frac{1}{N - m - 2} \sum_{n=m+1}^{N-1} \sigma_n^2 \tag{3.10}$$

has the minimum [77].

Fig. 3.2 shows the method application to time series of the Mackey-Glass equation [78]

$$\dot{x}(t) = -bx(t) + \frac{ax(t - \tau)}{1 + x^c(t - \tau)}, \tag{3.11}$$

which can be converted to Eq. (3.5) by division by b. The parameters of Eq. (3.11) are chosen to be $a = 0.2$, $b = 0.1$, $c = 10$, and $\tau = 300$ to produce a dynamics on a chaotic attractor. The sampling time is $\Delta t = 1$ and the number of points is $N = 10000$. Part of the time series is shown in Fig. 3.2(a). Fig. 3.2(b) depicts the dependence of D on the trial delay time m for two different numbers k of nearest neighbors for vector $\vec{X}_n = (x_n, x_{n-m})$. The value of m is varied from 1 to 500 with a step of 1. Both dependences $D(m)$ exhibit a well-pronounced absolute minimum at $m = 300$, which provides an accurate recovery of the discrete delay time $d = \frac{\tau_1}{\Delta t} = 300$.

The method efficiency was tested in the presence of noise. The location of the minimum of $D(m)$ allowed us to recover the delay time accurately even when a zero-mean Gaussian white noise added to the system time series had a standard deviation of 30% of the standard deviation of the data without noise (the signal-to-noise ratio was about 10 dB). Such level of noise greatly exceeds the noise level that is allowed for applying most of other methods of delay time reconstruction. The method can be extended to time-delay systems (3.1) of high order and with several coexisting delays [77].

A separate group of methods for the reconstruction of time-delay systems is based on the analysis of a system's response to external perturbations [79–81]. These methods are especially useful for the recovery of time-delay system performing periodic oscillations, since from periodic time series it is not possible to define whether the system is governed by delay-differential equation or ordinary differential equation and thus, one cannot recover the delay time. To solve the problem of delay time estimation in this case it was proposed to disturb the system by a short-correlated noisy signal [79], a control signal suitably designed to drive the system to a steady state [80], or a periodic impulsive signal leading to the appearance of a transient process [81]. All these methods [79–81] require sufficiently large amplitude of perturbations. For example, in [80] the amplitude of the signal of perturbation was by order of magnitude greater than the amplitude of unperturbed self-sustained oscillations. However, the use of strong disturbances

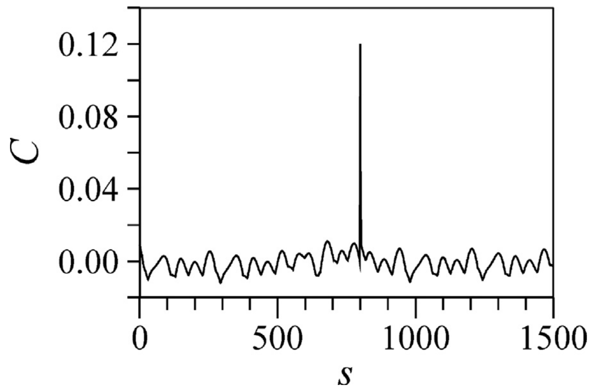

Fig. 3.3. The cross-correlation function (3.12) for the system (3.5) disturbed by rectangular pulses.

of a time-delay system is not always possible because it can result in undesirable qualitative change of the system behavior. In these cases, the use of small disturbances for estimating the system parameters is preferable. Such technique based on investigation of the cross-correlation function of the signals of perturbation and the system response has been proposed in [81]. For extracting the response of the system to small periodic signal of disturbance we used the method of accumulation [82].

Another efficient method for the reconstruction of time-delayed feedback systems using small disturbances has been proposed in [83]. This method is based on the analysis of the system response to a weak external disturbance having the form of rectangular pulses. To recover the delay time τ_1, we calculated the cross-correlation function of $\ddot{x}(t)$ and the second derivative of perturbation $\ddot{y}(t)$:

$$C(s) = \frac{< |\ddot{y}(t)||\ddot{x}(t+s)| >}{\sqrt{< |\ddot{y}(t)|^2 >< |\ddot{x}(t)|^2 >}}, \tag{3.12}$$

where the angular brackets denote averaging over time [83].

Fig. 3.3 shows the method application to the system (3.5) with $\tau_1 = 800$, $\varepsilon_1 = 20$, $f(x) = \lambda - x^2$, and $\lambda = 1$ disturbed by an external signal $y(t)$ having the form of rectangular pulses with amplitude $A = 0.01$, period $T = 1900$, and duration $M = \frac{T}{2}$. At these parameter values, the system (3.5) performs periodic oscillations in the absence of disturbance. The derivatives $\ddot{x}(t)$ and $\ddot{y}(t)$ were estimated from the time series of $x(t)$ and $y(t)$ using the simplest difference method. To construct the plot of $C(s)$ [Fig. 3.3] we used 20000 points, but the method can be applied to shorter time series. $C(s)$ has a pronounced maximum at $s = \tau_1$. As the length of the time series decreases, the maximum of $C(s)$ at $s = \tau_1$ becomes less pronounced. For the indicated parameter values it is sufficient to take only 3500 points, i.e., the use of two pulses is sufficient for the accurate reconstruction of τ_1. Note that for the accurate recovery of τ_1 the time series of $x(t)$ and $y(t)$ should be sampled at least at $\frac{\tau_1}{100}$.

Thus, the method allows one to use very short and low-amplitude pulses. It can be successfully applied to short time series and data heavily corrupted by noise.

4. Coupled phase oscillators

In order to detect and characterize couplings between oscillatory systems, one often relies on phases of oscillations (e.g. [84-106]) because of sensitivity of the phase as a dynamical variable to weak influences on a system (e.g. [85,88]). In particular, one is interested in detection of coupling in general, detection of coupling in a given direction, quantifying coupling "strength", and estimation of coupling time delays. These questions were considered by many authors (e.g. [84-106]), while our team addressed them [107-127] with an accent on assessment of statistical significance of conclusions.

4.1. Detection of coupling between oscillators

To detect coupling from oscillation phases $\varphi_1(t)$ and $\varphi_2(t)$, one uses different indices of phase synchronization [85], the most widely known among them being mean phase coherence [87] $< \rho = |e^{i(\varphi_1(t)-\varphi_2(t))}| >$ where angle brackets denote expectation. Nonzero estimate of ρ is a sign of coupling presence, while statistical significance is checked with the use of surrogate data or analytic formulas [93,95] assuming that oscillators do not possess individual phase nonlinearity and phase noises are white. In [122,123], we have suggested a more generally applicable approach based on correlation of phase increments $r = \frac{<(\Delta\varphi_1 - w_1)(\Delta\varphi_2 - w_2)>}{\sigma_{\Delta\varphi_1}\sigma_{\Delta\varphi_2}}$ where $\Delta\varphi_k(t) = \varphi_k(t+\tau) - \varphi_k(t)$ are increments over an interval τ, $w_{1,2} =< \Delta\varphi_{1,2} >$ are expectations, and $\sigma_{\Delta\varphi_1}, \sigma_{\Delta\varphi_2}$ are standard deviations of those increments. An estimator of r from a time series is suggested [122,123] to be the sample correlation coefficient $\hat{r}$. For a long enough time series, the estimator $\hat{r}$ is normally distributed with expectation r and variance given by Bartlett's formula [128]. Having that estimate of variance $\hat{\sigma}_{\hat{r}}^2$, one gets 95% confidence band for r as $\hat{r} \pm 1.96 \cdot \hat{\sigma}_{\hat{r}}$: A positive conclusion (i.e. that about coupling presence) is made from a given time series at a significance level of 0.05 if $|\hat{r}| > 1.96 \cdot \hat{\sigma}_{\hat{r}}$.

For numerical illustration, phase oscillators are used in Refs. [122] in the form

$$\begin{aligned} \frac{d\varphi_1}{dt} &= \omega_1 + b\sin\varphi_1 + k_{d,1}\sin(\varphi_2 - \varphi_1) + k_m \sin\varphi_2 + \xi_1(t), \\ \frac{d\varphi_2}{dt} &= \omega_2 + b\sin\varphi_2 + k_{d,2}\sin(\varphi_1 - \varphi_2) + k_m \sin\varphi_1 + \xi_2(t), \end{aligned} \tag{4.1}$$

where ω_1 and ω_2 are angular frequencies, b is phase nonlinearity parameter, $k_{d,1}, k_{d,2}$ are coefficients of "difference coupling", k_m is coefficient of "modulating coupling", phase noises ξ_1 and ξ_2 are mutually independent with covariance functions $\xi_k(t)\xi_k(t') = D_k\delta(t-t'), k = 1, 2$, where δ is Dirac delta function, D_1 and D_2 are noise intensities. Pure difference ($k_m = 0$) and pure modulating ($k_{d,1} = k_{d,2} = 0$) coupling have been considered. The difference coupling has been considered with $b = 0$ in "symmetric" ($k_{d,1} = k_{d,2} = k_d$) and "anti-symmetric" ($k_{d,1} = -k_{d,2} = k_d$) form. In the former case, coupling is synchronizing (for zero noises and small frequency mismatch, phase synchronization regime 1:1 gets stable for $k_d > |\omega_1 - \omega_2|/2$). In the latter case, coupling is non-synchronizing. The modulating coupling has been considered for "linear" ($b = 0$) and "nonlinear" ($b \neq 0$) oscillators. Fig. 4.1 provides estimation results for ensembles consisting of $M = 100$ pairs of time series with approximately 20 data points over basic period and of the length about 100 basic periods: frequency of positives f is shown along with ensemble-averaged values $<\hat{r}>$ for $\omega_1 = 1.1$, $\omega_2 = 0.9$, $D_1 = 0.04$, $D_2 = 0.01$. These results evidence that the method works properly, since the frequency of false positives is no greater than 0.05 (the dashed lines for $k_d = 0$ and $k_m = 0$). Besides, the method is quite sensitive to the difference synchronizing coupling: f gets large for small (as compared to $(\omega_1 - \omega_2)/2 = 0.1$) values of k_d and rises with k_d (Fig. 4.1,a). Even faster rise of f occurs for the difference anti-symmetric coupling (Fig. 4.1,b). Sensitivity to the modulating coupling is also high enough (Fig. 4.1,c,d), especially in the presence of phase nonlinearity (Fig. 4.1,d).

Mean phase coherence ρ is not sensitive to modulating and difference anti-symmetric couplings [122,123]. For difference synchronizing coupling, absolute values of r and ρ increase with k approximately at the same rate. Only unidirectional synchronizing coupling exhibits a clear advantage of ρ in terms of sensitivity. Thus, the suggested method based on r is simple in its implementation, does not require large amount of computations (as

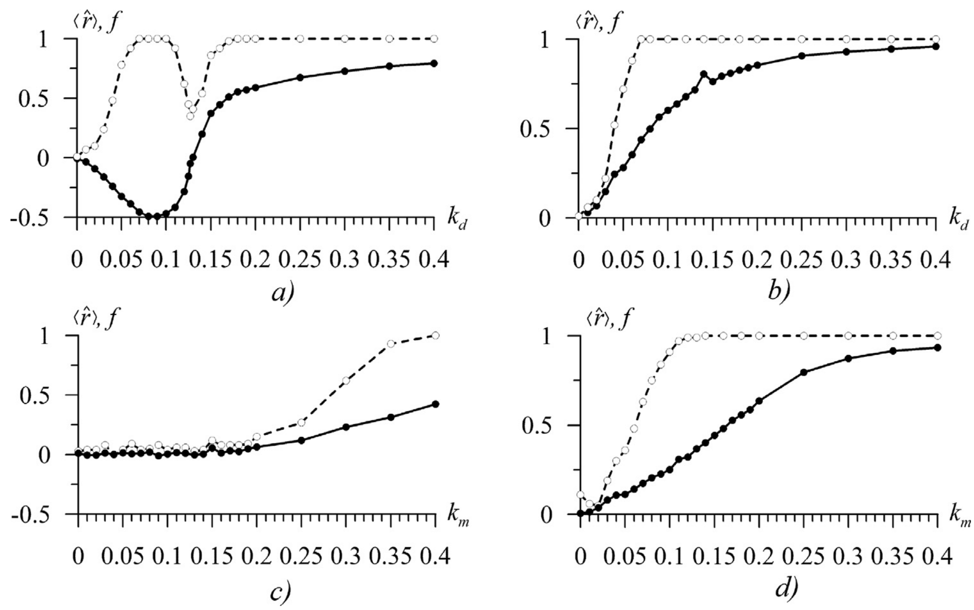

Fig. 4.1. Mean values $<\hat{r}>$ (solid lines) and frequencies of positives f (dashed lines): a) difference coupling with $k_{d,1} = k_{d,2} = k_d$, b) difference coupling with $k_{d,1} = -k_{d,2} = k_d$, c) modulating coupling at $b = 0$, d) modulating coupling at $b = 0.7$.

compared to surrogate data-based one), and widely applicable (as compared to ρ with analytic formulas for significance [93,95]).

4.2. Detection of coupling in a given direction

In nonlinear dynamics, one of the pioneering works on empirical estimation of directional couplings was that of Rosenblum and Pikovsky [89]. Its basic idea is to find out how strongly future evolution of one phase depends on another phase given the current value of the former phase. To implement this idea, one constructs a mathematical model from a time series of phases of two oscillatory systems in the form [89,107]

$$\Delta\varphi_k(t) = F_k(\varphi_1(t), \varphi_2(t), \mathbf{a}_k) + \varepsilon_k(t),\ k = 1,\ 2, \tag{4.2}$$

where $\Delta\varphi_k(t) \equiv \varphi_k(t+\tau) - \varphi_k(t)$, τ is a certain finite interval, $\varepsilon_k(t)$ are zero-mean noises, F_k are trigonometric polynomials and $\mathbf{a}_k$ are vectors of their coefficients. Its underpinning is a universal model of phase dynamics in the form of stochastic differential equations [85,86]

$$d\varphi_k/dt = \omega_k + G_k(\varphi_1, \varphi_2) + \xi_k(t),\ k = 1,\ 2, \tag{4.3}$$

where ω_k govern angular frequencies of oscillations, $\xi_k(t)$ are white noises. To construct a model (4.2), one specifies τ, often equal to a basic period of oscillations [89], and orders of the polynomials F_k. In many works of our team (e.g. [107-115]), polynomials of the third order are used following Ref. [89]. Estimates of coefficients $\hat{\mathbf{a}}_k$ are obtained via the least-square technique, i.e. via

$$\sigma^2_{\varphi,k} = \frac{1}{N-\tau}\sum_{i=1}^{N-\tau}(\Delta\varphi_k(t_i) - F_k(\varphi_1(t_i), \varphi_2(t_i), \mathbf{a}_k))^2 \to \min, k = 1, 2. \tag{4.4}$$

Estimates of the influences of oscillators on each other are obtained from the estimates $\hat{\mathbf{a}}_k$.

For a priori known equations of phase dynamics, the strength of influence of the second system on the first one c_1 was defined as steepness of the dependence of F_1 on φ_2 [89,107], so

$$c^2_{1,2} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(\partial F_{1,2}(\varphi_1, \varphi_2, \mathbf{a}_{1,2})/\partial\varphi_{2,1})^2 d\varphi_1 d\varphi_2. \tag{4.5}$$

To obtain estimates of c_1 and c_2 from a time series, one can just substitute $\hat{\mathbf{a}}_k$ for the true values $\mathbf{a}_k$ into the formula (4.5). However, such estimates $\hat{c}_{1,2}$ may often be strongly biased, even for moderately long time series of hundreds of basic periods as shown in [107]. In Ref. [107], we have suggested alternative time series-based estimators $\hat{\gamma}_{1,2}$ for the quantities $c^2_{1,2}$ and obtained formulas for their 95% confidence bands as $[\hat{\gamma}_k - 1.6\hat{\sigma}_{\hat{\gamma}_k}, \hat{\gamma}_k + 1.8\hat{\sigma}_{\hat{\gamma}_k}]$, where $\hat{\sigma}_{\hat{\gamma}_k}$ are computed from the same time series. The estimators $\hat{\gamma}_{1,2}$ are unbiased under sufficiently general conditions and the confidence bands assure the frequency of false positives not greater than 0.05 for time series not shorter than 50 basic periods [107]. Applicability of the estimators is often retained for shorter periods down to the length of 20 basic periods, if the sample mean phase coherence $\hat{\rho}$ does not exceed 0.4 [109]. More detailed and rigorous conditions of applicability are obtained in Ref. [113]. Different numerical examples are given in Refs. [107,109,113].

4.3. Quantifying coupling strength

Apart from detection of coupling presence, it is often desirable to have more vivid and conceptually interpreted characteristics of "coupling strength". Refs. [117,119] are devoted to obtaining such quantifiers. As indicated above, phase dynamics of weakly coupled self-oscillatory systems is described accurately enough with Eq.s (4.2), if contributions of the noises and couplings to phase increments $\Delta\varphi_k(t) = \varphi_k(t+\tau) - \varphi_k(t)$ are small in comparison with the "linear increment" $\omega_k\tau$. Then, the strength of the influence of the j-th oscillator on the k-th one ($j \to k$) is defined in [117,119] without derivatives, in a simpler form. In the simplest case of first-order coupling $F_k = \alpha_k \sin(\varphi_j - \varphi_k) + \beta_k \cos(\varphi_j - \varphi_k)$, consider statistical properties of the phase increment $\Delta\varphi_k$ in the equation (C2). Its expectation is $< \Delta\varphi_k >= w_k \approx \omega_k\tau$. Due to noise and coupling from another oscillator, the quantity $\Delta\varphi_k$ fluctuates about its expectation, representing modulation of the oscillation period. Under weak enough coupling, stationary probability distribution of wrapped phases ($\varphi_1 \bmod 2\pi, \varphi_2 \bmod 2\pi$) is almost uniform over the square $[0, 2\pi) \times [0, 2\pi)$. The functional terms entering F_k are mutually orthogonal over that domain. So, averaging both squared sides of the Eq. (4.2) shows that the variance of $\Delta\varphi_k$ is a sum

$$\sigma^2_{\Delta\varphi_k} = c_{j\to k} + \sigma^2_{\varepsilon_k}, \tag{4.6}$$

where $\sigma^2_{\varepsilon_k} = \sigma^2_{\xi_k}\tau$ is the variance of the noise $\varepsilon_k(t)$ and the quantity

$$c_{j\to k} = \frac{1}{2}\left(\alpha_k^2 + \beta_k^2\right) \quad (4.7)$$

can be called “strength” of the influence $j \to k$. Such decomposition of phase increment fluctuation intensity provides the coupling strength $c_{j\to k}$ with a clear “oscillatory” sense. Having the least-squares estimates $\hat{w}_k, \hat{\alpha}_k, \hat{\beta}_k$ of the coefficients of the model (4.2), one estimates $c_{j\to k}$ as $\hat{c}_{j\to k} = \frac{1}{2}(\hat{\alpha}_k^2 + \hat{\beta}_k^2)$. Under the hypothesis of zero coupling ($c_{j\to k} = 0$), the random quantities $\hat{\alpha}_k$ and $\hat{\beta}_k$ are mutually independent and identically normally distributed with zero mean and some variance $\sigma^2_{\hat{\alpha}_k}$, so the quantity $\chi^2_{j\to k} = \frac{\hat{\alpha}_k^2+\hat{\beta}_k^2}{\sigma^2_{\hat{\alpha}_k}}$ is distributed via the chi-square law with two degrees of freedom. Denote (1-p)-percentile of this distribution as $\hat{\chi}^2_{2,1-p}$, which is such number that $\Phi_2(\hat{\chi}^2_{2,1-p}) = 1-p$. If one gets $\chi^2_{j\to k} > \hat{\chi}^2_{2,1-p}$, then the hypothesis of zero coupling can be rejected at the significance level p. The value of $\sigma^2_{\hat{\alpha}_k}$ can be replaced with its estimate derived in Ref. [107] under the assumption that covariance function of the noise ε_k decreases linearly down to zero over the interval $[0, \tau]$.

Everything is analogous for higher-order polynomials F with $\boldsymbol{a}_k = (w_k, \{\alpha_{k,m,n}, \beta_{k,m,n}\}_{(m,n)\in\Omega_k})$ where Ω_k is the summation range, i.e. the set of integers m and n determining which terms enter the polynomial. The strongest influence on dynamics of the kth oscillator is exerted by the resonance terms, i.e. those with $m/n \approx \omega_j/\omega_k$. Still, non-resonant terms can also be important [129]. Strength of coupling $j \to k$ is defined in Ref. [119] in analogy to (4.7) based on the equation

$$< (\Delta\varphi_k)^2 >= w_k^2 + \frac{1}{2}\sum_{(m,n)\in\Omega_k}\left(\alpha^2_{k,m,n} + \beta^2_{k,m,n}\right) + \sigma^2_{\varepsilon_k}. \quad (4.8)$$

The terms in (C8) with $n \neq 0$ specify the influence $j \to k$. Their sum can be called its strength:

$$c_{j\to k} = \frac{1}{2}\sum\nolimits_{(m,n)\in\Omega_k,\ n\neq 0}\left(\alpha^2_{k,m,n} + \beta^2_{k,m,n}\right). \quad (4.9)$$

The remaining terms specify individual nonlinearity of the phase dynamics $\mathrm{b_k}$:

$$b_k = \frac{1}{2}\sum\nolimits_{(m,n)\in\Omega_k,\ n=0}\left(\alpha^2_{k,m,n} + \beta^2_{k,m,n}\right). \quad (4.10)$$

Since $< \Delta\varphi_k >= w_k$, the variance of the phase increment $\sigma^2_{\Delta\varphi_k} =< (\Delta\varphi_k)^2 > - < \Delta\varphi_k >^2$ reads

$$\sigma^2_{\Delta\varphi_k} = b_k + c_{j\to k} + \sigma^2_{\varepsilon_k}. \quad (4.11)$$

The quantity $c_{j\to k}$ can be further normalized in different ways [119] to have different interpretations, e.g. $c_{j\to k}/w_k^2$ shows intensity of oscillation period modulation due to the influence $j \to k$ and $c_{j\to k}/\sigma^2_{\Delta\varphi_k}$ provides “percentage” of intensity of the phase increment fluctuations determined by the influence $j \to k$. Such coupling strength depends on the selected time scale τ and relative roles of the noise and coupling differ with the time scale. It appears practically reasonable to select τ to be about the basic oscillation period T [107]. Estimator of that coupling strength reads

$$\hat{c}_{j\to k} = \frac{1}{2}\sum\nolimits_{(m,n)\in\Omega_k,\ n\neq 0}\left(\hat{\alpha}^2_{k,m,n} + \hat{\beta}^2_{k,m,n}\right). \quad (4.12)$$

Recall that the sum of squares of M independent quantities normally distributed with zero mean and unit variance is distributed as chi-square with M degrees of freedom. So, the quantity $\chi^2_{j\to k} = \sum_{m,n\ (n\neq 0)}\frac{\hat{\alpha}^2_{k,m,n}+\hat{\beta}^2_{k,m,n}}{\sigma^2_{\hat{\alpha}_{k,m,n}}}$ is distributed as chi-square with M_k degrees of freedom where M_k is the number of terms in the right-hand side of (4.12). The value of $\chi^2_{j\to k}$ can be computed from a time series if the variance $\sigma^2_{\hat{\alpha}_{k,m,n}}$ is replaced with its estimate derived in Ref. [107]. These estimates of coupling strengths and assessment of their confidence apply under some conditions such as well-defined phases and low mean phase coherence of the respective order, etc [119].

Applicability of the suggested estimates for moderate lengths of time series is shown in numerical experiments with exemplary oscillators with different properties of phase dynamics [119]. Generalization of the approach to larger ensembles of oscillators is straightforward and has been accomplished in Ref. [117] with successful results for ensembles consisting of 10 exemplary oscillators and even for a discrete scheme for a continuously distributed system.

4.4. Estimation of the coupling delay time

To determine time delay of the coupling from the jth oscillator to the kth one, it was suggested in [130] to build a model in the form

$$\varphi_k(t+\tau) - \varphi_k(t) = F\left(\varphi_k(t), \varphi_j(t-\Delta)\right) + \varepsilon_k(t), \quad (4.13)$$

where the trial delay Δ is a free parameter too. Coefficients of F are estimated via minimization of the mean-squared error $S(\Delta) = \hat{\varepsilon}_k^2(t_i)$ where $\hat{\varepsilon}_k(t_i) = \varphi_k(t_i+\tau) - \varphi_k(t_i) - F_k(\varphi_k(t_i), \varphi_j(t_i-\Delta))$, and then S is minimized over Δ. In our works, to account for possible biases and random errors of statistical estimation, this approach was further elaborated [121]. First, a point estimate of the coupling delay unbiased under general conditions is derived to be $\hat{\Delta} = \Delta_{min} + \tau/2$ where $\Delta_{min} = \arg\min\limits_{\Delta} S(\Delta)$. The variance of $\hat{\Delta}$ under the assumption of linear decrease of the covariance function of ε_k down to zero over an interval of lags $[0,\tau]$ is estimated as

$$\hat{\sigma}^2_{\Delta} = \frac{2\hat{\sigma}^2_{\varepsilon}}{N'}\left(\left.\frac{d^2S(\Delta)}{d\Delta^2}\right|_{\Delta=\Delta_{\min}}\right)^{-1}, \quad (4.14)$$

where $N' = N\Delta t/\tau$ is the number of non-overlapping intervals of the length τ inside a time series, $\hat{\sigma}^2_{\varepsilon_k} = \min\limits_{\Delta} S(\Delta)$ is an estimate of the variance of ε_k. To estimate the second derivative in (4.14), the dependence $S(\Delta)$ is approximated in the vicinity of the minimum point $\Delta_{\min}$ with a quadratic parabola. An interval estimator of the coupling delay (its 95% confidence band) is $\hat{\Delta} \pm 1.96\hat{\sigma}_{\Delta}$. Efficiency of the method was shown for phase oscillators and Van der Pol oscillators driven by white noise [121]. Further refinements were developed in Ref. [124,125] for other properties of noise, namely for noises with longer correlations such as those in the phase oscillators of the form

$$\begin{aligned}
&d\varphi_1(t)/dt = \omega_1 + \xi_1(t),\\
&d\xi_1(t)/dt = -\alpha_1\xi_1(t) + \eta_1(t),\\
&d\varphi_2(t)/dt = \omega_2 + k\sin(\varphi_1(t-\Delta_0) - \varphi_2(t)) + \xi_2(t),\\
&d\xi_2(t)/dt = -\alpha_2\xi_2(t) + \eta_2(t),
\end{aligned} \quad (4.15)$$

where $\omega_{1,2}$ are individual angular frequencies, k is a coupling coefficient, Δ_0 is a coupling delay (for the influence $1 \to 2$), $\eta_{1,2}$ are mutually independent white noises with covariance functions $\eta_k(t)\eta_k(t') = D_k\delta(t-t')$, and so $\xi_{1,2}$ are colored noises which can be called “frequency noises”. The variance of ξ_k is expressed via the white noise η_k intensity as $\sigma^2_{\xi_k} = D_k/(2\alpha_k)$. In Refs. [124,125] it is shown that for $\omega_1 = 1.05$, $\omega_2 = 0.95$, $k = 0.1$, $\Delta_0 = 12$ (40 data points), $\sigma_{\xi_2} = 0.06$, $\alpha_1 = 0.11$, and $\alpha_2 = 0.09$, the rate of false conclusions about the time delay value gets large (greater than 0.05, and even than 0.1) at $\sigma_{\xi_1} < 0.17$. To diagnose that situation, one can use the sample autocorrelation function of the noise ε_2 which diminishes for time lags much larger than τ. Such properties of the phase dynamics are encountered also for low-dimensional nonlinear oscillators in a deterministically chaotic regime where the use of Eq. (4.13) without explicitly included amplitude dynamics is not rigorously justified but often appears a good approximation with the properties of phase noise determined by the influence of slowly varying chaotic amplitudes.

In Ref. [124,125] it is suggested to use the sample autocorrelation function of ε_k estimated from the model residual errors and determine its decay time T as the time lag at which it decreases down to a small value, e.g. empirically selected value of 0.2 suffices for time series of the length about 100 basic periods. Estimating the number of independent values of ε_k in a time series from below as $N'' = N\Delta t/L$ where $L = \max[T, \tau]$, an estimator of the variance of $\hat{\Delta}$ takes the form

$$\hat{\sigma}_{\Delta}^2 = \frac{2\hat{\sigma}_{\varepsilon}^2}{N''}\left(\left.\frac{\partial^2 S(\Delta)}{\partial \Delta^2}\right|_{\Delta=\hat{\Delta}_{\min}}\right)^{-1}, \tag{4.16}$$

which gives a wider interval than Eq. (4.13) for $T > \tau$. This idea appears efficient, e.g. the noise variance in the driving oscillator $\sigma_{\xi_1}^2$ for a system (4.15) was varied in a wide range via changing D_1 at other fixed parameters in Ref. [124,125], in particular for $\alpha_1 = 0.11, \alpha_2 = 0.09$ (long correlations) and $\alpha_1 = 11, \alpha_2 = 9$ (quickly decaying correlations, i.e. almost white phase noise). The rate of false conclusions about the delay value with the estimate (4.14) is large at $\sigma_{\xi_2} = 0.06$, $\alpha_1 = 0.11, \alpha_2 = 0.09$ and sufficiently small $\sigma_{\xi_1} < 0.2$. An elaborated estimator (4.16) allows one to reduce the errors rate down to the fixed small value of 0.05. Still, for $\alpha_1 = 11, \alpha_2 = 9$ and small σ_{ξ_1}, even Eq. (4.16) does not work since S appears to be too weakly sensitive to the trial delay Δ and so random fluctuations strongly shift the location of the minimum of $S(\Delta)$. Diagnosis of this problematic situation can be done on the basis of the plot $S(\Delta)$ which does not possess a single main minimum as distinct from a "good" situation of clear minimum. Analogous results were obtained for Van der Pol oscillators with the phases determined via the Hilbert transform [125].

An example of coupled low-dimensional nonlinear systems with possible periodic and chaotic dynamics with well-defined phase is given by the Roessler systems:

$$\begin{aligned}
\dot{x}_1(t) &= -\omega_1 y_1(t) - z_1(t) + \xi_1, \\
\dot{y}_1(t) &= \omega_1 x_1(t) + a y_1(t), \\
\dot{z}_1(t) &= b - z_1(t)(r - x_1(t)), \\
\dot{x}_2(t) &= -\omega_2 y_2(t) - z_2(t) + K(x_1(t-\Delta_0) - x_2(t)) + \xi_2, \\
\dot{y}_2(t) &= \omega_2 x_2(t) + a y_2(t), \\
\dot{z}_2(t) &= b - z_2(t)(r - x_2(t)),
\end{aligned} \tag{4.17}$$

where $\omega_1 = 1.015, \omega_2 = 0.985$ are angular frequencies, $a = 0.1$, $b = 0.1$, and parameter r is varied in a wide range providing different dynamical regimes in Ref. [126,127] from a cycle of period one via period-doubling cascade to chaos, $\xi_{1,2}$ are white noises with intensities $D_{1,2}$, coupling delay is $\Delta_0 = 12$, K is coupling coefficient. Phase dynamics of an individual system is described with a sufficiently cumbersome equation demonstrating that even for nonzero noises one must in general introduce into the phase dynamics model (4.13) additional "noise terms" representing the influence of "the amplitude on the plane x-y" and of the third coordinate z. Such terms are sometimes called "efficient noises" and properties of such "noises" for low-dimensional chaotic systems may be rather non-trivial leading to difficulties in applying an asymptotic estimator (4.16). At nonzero $D_{1,2}$ in the system (4.17), noises in the phase models (4.13) approximate a combined influence of random processes $\xi_{1,2}$ and unaccounted dynamical variables of the original system. Estimation in numerical experiments was performed in Refs. [126,127] at $K = 0.05$ for "individually" periodic ($r = 4$) and chaotic ($r = 10$) regimes. "Good" situations (with the errors

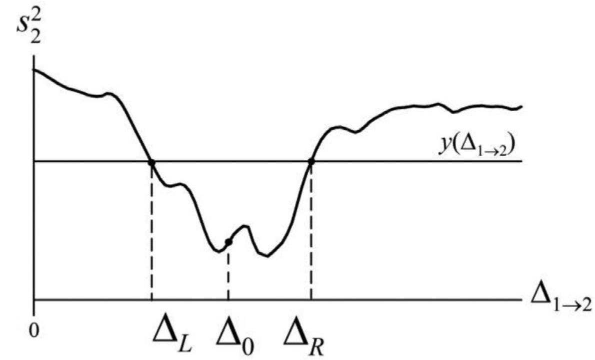

Fig. 4.2. Illustration for the rough estimator of the coupling delay.

rate less than 0.1) include a perturbed periodic regime at $D_1 > 0.5$ and perturbed chaotic regime at $D_1 > 0.35$. At lower noise levels, the error rate exceeds 0.1. Seemingly, a nonzero bias of the delay estimator is determined by peculiarities of interaction between the phase and other variables unaccounted in the model (4.13) that leads to inadequacy of the phase description with independent external phase noises (4.13). Such situations differ from the case of the original system (4.3) with white noises by the following circumstance: Even if the plot $s_2^2(\Delta_{1\to2})$ exhibits a clear main minimum, its shape is not close to a quadratic parabola or sufficiently deep local minima arise apart from the global one. These features are seemingly determined by an individual character of the concrete system nonlinearity.

To overcome these problems, Ref. [127] suggests to refuse from a local approximation of the plot with a parabola and to use a rough estimate of the global minimum width. Namely, one can draw a straight line on the plane $(\Delta_{1\to2}, s_2^2)$ parallel to the abscissa axis at the ordinate level equal to the middle between the minimal and maximal values of s_2^2 in the selected range of trial $\Delta_{1\to2}$. As an interval estimator of the coupling delay, one can take an interval $[\Delta_L, \Delta_R]$ between the leftmost and rightmost cross-section points (of the drawn line and the plot $s_2^2(\Delta_{1\to2})$) (Fig. 4.2).

In estimating a coupling delay with the rough interval estimator, the errors rate appears to be less than 0.05 in all above mentioned problematic situations [127]. Exceptions are those cases at small values of D_1 when the plot $s_2^2(\Delta_{1\to2})$ does not possess a single clear minimum, but they can be readily diagnosed in practice. Higher reliability of the rough estimator is achieved at the expense of broadening the confidence band, i.e. reducing sensitivity of the estimator as a detector of small nonzero delays. However, even the rough estimator is often sufficiently sensitive and can detect nonzero delays when an asymptotic estimator (C16) provides false conclusions about the coupling delay. The rough method appears efficient even for a number of systems with difficulties in definition of the phase [127] when there is no clear rotation of an orbit about a single center on a complex plane, either due to random influences and relatively wide power spectrum (e.g. for linear stochastic oscillators) or peculiarities chaotic dynamics and large phase diffusion (e.g. for Lorenz systems).

5. Conclusions

Special approaches to reconstruction of dynamics equations involving prior knowledge about a system under study appear promising in practice. Here, we have overviewed some directions in this field which include coping with hidden variables, reconstruction of time-delay systems, and analysis of couplings between oscillatory systems based on phase dynamics modeling. We have presented a series of results of our team and only briefly mentioned many contributions of other groups actively working in the field of time series analysis and reconstruction of dynamics.

To summarize, parameter estimation in fully known model structure under the hidden variables setting is useful to verify conceptual models and make more accurate estimates of their parameters. Such problem setting is quite widespread and interesting in practice, e.g. [24,25]. Our works contribute to making the estimation techniques more efficient and promise some further improvements [27,42,44,45].

We have proposed several methods for the reconstruction of delay times for various classes of time-delayed feedback systems from their time series. The method based on the statistical analysis of time intervals between extrema in the time series is very simple. This method uses only operations of comparing and adding. It needs neither ordering of data, nor calculation of approximation error or certain measure of complexity of the trajectory and therefore, it is quick-operating. The method based on the nearest neighbour analysis needs more time of computation. However, it remains efficient under very high levels of dynamical and additive noise. The method based on the analysis of a time-delay system response to an external disturbance having the form of rectangular pulses is intended for the reconstruction of delay times in the case of periodic time series. This method allows one to use very short and low-amplitude pulses.

Phase dynamics-based methods of coupling analysis represent quite universal tool for studying oscillatory systems with pronounced main rhythms, e.g. [89,91,107]. Such phase-dynamical methods developed in the works overviewed here are suitable to reveal couplings and their delays from sufficiently short time series for oscillatory systems with diverse properties. With the aid of these methods, we have obtained a number of new results from climate and neurophysiological data as presented in Refs. [109-112,118] (climate) and Refs. [115,116,120] (neurophysiology).

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Credit authorship contribution statement

B.P. Bezruchko: Supervision, Resources, Funding acquisition, Writing - original draft. V.I. Ponomarenko: Methodology, Validation, Writing - original draft. D.A. Smirnov: Investigation, Software, Validation, Visualization, Writing - original draft. I.V. Sysoev: Investigation, Software, Validation, Visualization, Writing - original draft. M.D. Prokhorov: Project administration, Writing - original draft, Writing - review & editing.

Acknowledgements

This work was carried out within the framework of the state task. We thank our colleagues who contributed to the works presented above: A.S. Karavaev, E.V. Navrotskaya, M.V. Sysoeva, V.S. Khorev, and D.D. Kulminskiy.

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